SOLUTION - BANK

PEYAM RYAN TABRIZIAN

This document contains solutions to various Math 1A homework problems I've compiled, from Fall 2010 and Spring 2011. At first, the solutions might seem disappointingly short, but don't worry! As the course progresses, the solutions become longer and more detailed! It contains solutions to the following problems:

Section	Problems
1.1	6, 18, 32, 45, 57, 61
1.2	2, 4, 8, 16
1.3	1, 7, 14, 30, 36
1.5	3, 5, 6, 16, 17, 18
1.6	3, 5, 16, 18, 26, 35, 36, 39, 48, 58, 60, 64, 65, 66
2.2	2, 6, 28, 29, 40
2.3	4, 10, 13, 17, 18, 26, 29, 38, 47, 58, 59
2.4	2, 4, 19, 32, 37, 42, 43
2.5	3, 8, 9, 27, 34, 40, 47, 56, 65
2.6	4, 22, 26, 32, 34, 57
2.7	6, 12, 17, 18, 32, 34, 40, 46
2.8	3, 21, 38, 43, 52
3.1	11, 13, 20, 32, 35, 49, 54
3.2	15, 17, 33, 41, 47, 53
3.3	5, 13, 37, 39, 40, 51
3.4	9, 15, 39, 45, 49, 63, 66, 70, 83, 84, 90
3.5	3, 19, 27, 28, 30, 40, 46, 49, 67, 69
3.6	13, 21, 39
3.7	4, 10, 17, 24, 29
3.8	4, 9, 11, 19
3.9	5, 13, 15, 27, 38, 44
3.10	2, 11, 15, 21, 25, 35, 40
3.11	9, 11, 15, 21, 23(a), 26, 29(a)(b), 31, 39
4.1	6, 38, 39, 51, 57
4.2	6, 13, 18, 23, 29, 32, 34, 36
4.3	9, 13, 27, 43, 45, 51
4.4	3, 4, 11, 14, 15, 23, 27, 29, 40, 47, 49, 51, 56, 59, 71
4.5	11, 31, 41, 47
4.7	3, 11, 19, 21, 30, 53, 63
4.8	38
4.9	7, 24, 33, 39, 61, 74
	Continued on the next page

Date: Monday, June 20th, 2011.

Section	Problems
5.1	2, 5, 15, 19, 21, 26
5.2	11, 18, 21, 23, 30, 34, 36, 43, 44, 47, 51, 52, 67, 68
5.3	4, 7, 15, 17, 25, 31, 35, 41, 43, 54
5.4	10, 12, 13, 25, 37, 47, 52, 58, 59, 61, 62
5.5	7, 31, 39, 46, 59, 73, 88
6.1	1, 3, 13, 21, 40, 41, 49
6.2	3, 6, 13, 17, 49, 51, 57, 67, 70
6.3	2, 13, 15, 19, 44, 46

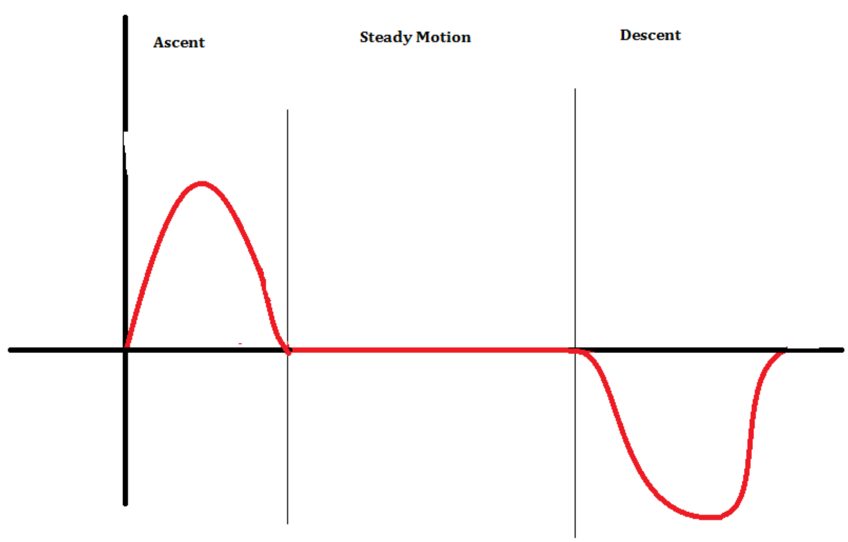
Section 1.1: Four ways to represent functions

1.1.6. Yes (by the vertical line test), Domain $= [-2, 2]$, Range $= [-1, 2]$

1.1.18.

(a) The graph of $x(t)$ should just be a line going through the origin
(b) The graph of $y(t)$ should look at first like the right half of a parabola, then should be constant for a while, and then look like the left half of a parabola
(c) The graph of the horizontal velocity looks like a horizontal line
(d) See announcement on bspace for a detailed solution! The picture you get is:

1A/Math 1A Summer/Solution Bank/Vertical Velocity.png



1.1.32. Domain = $[-2, 2]$, Range = $[0, 2]$, Graph is just the upper-half of the circle centered at 0 of radius 2.

1.1.45. $f(x) = \frac{5}{2}x - \frac{11}{2}$

1.1.57. $V(x) = x(20 - 2x)(12 - 2x)$ (no need to expand the answer!)

1.1.61. f is odd, g is even

Section 1.2: Mathematical models: a catalog of essential functions

1.2.2.

(a) Rational function
(b) Algebraic function
(c) Exponential function
(d) Power function
(e) Polynomial of degree 6
(f) Trigonometric function

1.2.4.

(a) G
(b) f
(c) F
(d) g

1.2.8. (a) $y = 2(x - 3)^2$, (b) $y = -x^2 - \frac{5}{2}x + 1$

1.2.16.

(a) $C(x) = 13x + 900$ (C is the cost and x is the number of chairs produced)
(b) 13; Cost per chair
(c) 900; Start-up cost (i.e. money needed to buy machines in order to *start* producing chairs)

Section 1.3: New functions from old functions

1.3.1.

(a) $y = f(x) + 3$
(b) $y = f(x) - 3$
(c) $y = f(x - 3)$
(d) $y = f(x + 3)$
(e) $y = -f(x)$
(f) $y = f(-x)$
(g) $y = 3f(x)$
(h) $y = \frac{1}{3}f(x)$

1.3.7. $y = -\sqrt{3(x + 4) - (x + 4)^2} - 1$

1.3.14. Basically compress the graph of $\sin(x)$ horizontally by a factor of 3 (notice that the new period now is $\frac{2\pi}{3}$ and then stretch the resulting graph vertically by a factor of 4 (so the new graph has range $[-4, 4]$ instead of $[-1, 1]$)

1.3.30.

(a) $(f+g)(x) = \sqrt{3-x} + \sqrt{x^2-1}$
(b) $(f-g)(x) = \sqrt{3-x} + \sqrt{x^2-1}$
(c) $(fg)(x) = \sqrt{3-x} \times \sqrt{x^2-1}$
(d) $\left(\frac{f}{g}\right)(x) = \frac{\sqrt{3-x}}{\sqrt{x^2-1}}$

All of those functions have domain $(-\infty, -1] \cup [1, 3]$ **EXCEPT** for (d), which has domain $(-\infty, -1) \cup (1, 3]$

1.3.36.

(a) $(f \circ g)(x) = \frac{\sin(2x)}{1+\sin(2x)}$; Dom = all odd multiples of $\frac{\pi}{2}$
(b) $(g \circ f)(x) = \sin\left(\frac{2x}{1+x}\right)$; Dom = all real numbers except -1
(c) $(f \circ f)(x) = \frac{\frac{x}{1+x}}{1+\frac{x}{1+x}} = \frac{x}{1+2x}$; Dom = all real numbers except $\frac{-1}{2}$ and -1
(d) $(g \circ g)(x) = \sin(2\sin(2x))$; Dom = all real numbers

Section 1.5: Exponential Functions

1.5.3. Basically, the larger the base, the faster the function is increasing

1.5.5. Notice that $\left(\frac{1}{3}\right)^x = 3^{-x}$, which means that $\left(\frac{1}{3}\right)^x$ is the reflection of 3^x across the y-axis! Similarly with 10^x.

1.5.6. The smaller the base, the faster the function is going to 0.

1.5.16. (a) All real numbers ; (b) All ≤ 0 real numbers

1.5.17. $f(x) = 3 \cdot 2^x$

1.5.18. $f(x) = 2 \cdot \left(\frac{1}{3}\right)^x = 2 \cdot 3^{-x}$

Section 1.6: Inverse functions and logarithms

1.6.3. No; For example, even though $2 \neq 6$, $f(2) = f(6) = 2$

1.6.5. Yes (by the horizontal line test)

1.6.16.

(a) $f^{-1}(3) = 0$
(b) $f(f^{-1}(5)) = 5$

1.6.18.

(a) By the horizontal line test
(b) Domain of f^{-1} = Range of $f = [-1, 3]$; Range of f^{-1} = Domain of $f = [-3, 3]$
(c) 0
(d) ≈ -1.8

1.6.26. $f^{-1}(x) = \ln\left(-\frac{x}{2x-1}\right) = \ln\left(\frac{x}{1-2x}\right)$

1.6.35.

(a) $\log_2(8) = 3$
(b) $\log_3\left(\frac{1}{9}\right) = -2$

1.6.36.

(a) $5^{-2} = \frac{1}{25}$
(b) 10

1.6.39. $\ln\left(\frac{(1+x^2)\sqrt{x}}{\sin(x)}\right)$

1.6.48.

(a) $x = \frac{\ln(7)-3}{2}$
(b) $x = \frac{5-e^{-3}}{2}$

1.6.58.

(a) $t = -a\ln\left(1 - \frac{Q}{Q_0}\right)$; Gives the time it takes to recharge the capacitor to a given capacity Q
(b) Plug in $Q = 0.9Q_0$ into the equation in (a), and you get $t = -2\ln(0.1) \approx 4.61$ seconds

1.6.60.

(a) $\frac{\pi}{6}$
(b) $\frac{\pi}{3}$

1.6.64.

(a) $\frac{\sqrt{15}}{4}$
(b) $\frac{24}{25}$ (use the fact that $\sin(2x) = 2\sin(x)\cos(x)$)

1.6.65. If $\theta = \sin^{-1}(x)$, then $\sin(\theta) = x$, then draw a triangle with hypothenuse 1, and opposite side x, and then the adjacent side becomes $\sqrt{1-x^2}$, and so our answer becomes:

$$\cos(\sin^{-1}(x)) = \cos(\theta) = \frac{adjacent}{hypotenuse} = \frac{\sqrt{1-x^2}}{1} = \sqrt{1-x^2}$$

See the handout ”Proof of the derivative of arccos” for a similar problem; Or look at your notes taken in section!

1.6.66. $\tan(\sin^{-1} x) = \frac{\sin(\sin^{-1}(x))}{\cos(\sin^{-1}(x))} = \frac{x}{\sqrt{1-x^2}}$ by the result of number 65!

Section 2.2: The limit of a function

2.2.2. If x approaches 1 from the left, then $f(x)$ approaches 3; If x approaches 1 from the right, then $f(x)$ approaches 7. No, left-hand-limits and right-hand-limits must be equal!

2.2.6.

(a) 4
(b) 4
(c) 4
(d) Undefined
(e) 1
(f) -1
(g) Does not exist (left and right-side limits not equal)
(h) 1
(i) 2
(j) Undefined
(k) 3
(l) Does not exist (h does not approach one fixed value as x approaches 5 from the left)

2.2.28. $-\infty$ (numerator approaches $e^{-5} > 0$ while denominator approaches 0^-

2.2.29. $-\infty$ ($x^2 - 9$ approaches 0^+ and $\ln(0^+) = -\infty$

2.2.40. The mass blows up to ∞ ($\frac{v^2}{c^2}$ goes to 1^-, so the denominator of the fraction goes to 0^+, and so the whole fraction goes to ∞)

Section 2.3: Calculating limits using the limit laws

2.3.4. $\frac{9}{12} = \frac{3}{4}$

2.3.10.

(a) If you plug in $x = 2$, then the left hand side is not defined, but the right hand side is
(b) The above equation holds if $x \neq 2$, but the point of limits is that in this case you don't **care** about the value at 2! So in this case, the equality is correct!

2.3.13. Does not exist (left-hand-limit is $-\infty$ because the numerator tends to 4 and the denominator tends to 0^- while the right-hand-limit is ∞ because the numerator tends to 4 and the denominator tends to 0^+)

2.3.17. 8

2.3.18. 3 (use the fact that $x^3 - 1 = (x-1)(x^2 + x + 1)$)

2.3.26. 1 (put under a common denominator $t^2 + t = t(t+1)$ and cancel out)

2.3.29. $\frac{1}{2}$ (put under a common denominator and multiply by the conjugate form)

2.3.38. 0 (by squeeze theorem, because $-1 \leq \sin\left(\frac{\pi}{x}\right) \leq 1$)

2.3.47.

(a)(i) 2 (since $|x-1| = x - 1$ in this case)
(a)(ii) -2 (since $|x-1| = 1 - x$ in this case)
(b) No, since the right-hand-limit and the left-hand-limit are not equal

2.3.58. Let $a = 0$ and $f(x) = \sin\left(\frac{1}{x}\right)$ (or $\frac{1}{x}$), and $g(x) = -f(x)$.

2.3.59. Let $a = 0$ and $f(x) = \sin\left(\frac{1}{x}\right)$ (or $\frac{1}{x}$), and $g(x) = \frac{1}{f(x)}$

Section 2.4: The precise definition of a limit

2.4.2. $\delta = 0.7$ (remember, the smaller the δ, the better!)

2.4.4. $\delta = 0.2$ (I picked this because $\left|\sqrt{0.5} - 1\right| \approx 0.28$ and $\left|\sqrt{1.5} - 1\right| \approx 0.22$, and just pick a number slightly smaller than both)

2.4.19. $\delta = 5\epsilon$

2.4.32. $\delta = \min\left\{1, \frac{\epsilon}{19}\right\}$

2.4.37. $\delta = min\left\{\frac{a}{2}, \frac{\epsilon}{\sqrt{a}\left(1+\frac{1}{\sqrt{2}}\right)}\right\}$

The next two are optional, but good for practice:

2.4.42. $\delta = \sqrt[4]{\frac{1}{M}}$

2.4.43. $\delta = e^M$ (where M is negative)

Section 2.5: Continuity

2.5.3. -4 (f not defined at -4; neither), -2 (left-hand-side and right-hand-side limits not equal; continuous from the left), 2 (ditto; continuous from the right), 4 (left-hand-side limit does not exist; continuous from the right)

2.5.8. This is my personal opinion, you might disagree with me

(a) Continuous
(b) Discontinuous (because of cliffs and skyscrapers)
(c) Discontinuous (you pay per mile as **whole**, it doesn't matter whether you've traveled 0.9 miles or 0.99 miles)
(d) Continuous

2.5.9. $g(3) = 6$

2.5.27. Continuous because it's a composition of ln (continuous) and a polynomial (continuous), Dom $= (-\infty, -1) \cup (1, \infty)$

2.5.34. $\tan^{-1}\left(\frac{2}{3}\right)$

2.5.40. Yes, you can check that the left-hand-side-limits and the right-hand-side limits are equal! Plug in values for G, M, R if you want to, for example $G = 2$, $M = 5$, $R = 7$

2.5.47. (For extra practice) Define $f(x) = x^4 + x - 3$, then $f(1) = -1 < 0$, $f(2) = 15 > 0$, so by IVT, there is one number c such that $f(c) = 0$.

2.5.56. Use the fact that $\sin(a + h) = \sin(a)\cos(h) + \sin(h)\cos(a)$

2.5.65. Define $f(t)$ to be the altitude of the monk on the first day, $g(t)$ to be the altitude of the monk on the second day, and let $h(t) = f(t) - g(t)$. Then $h(0) > 0$, $h(12) < 0$ (where 0 means $7AM$ and 12 means $12PM$), then by IVT, there is one number c such that $h(c) = 0$, i.e. $f(c) = g(c)$

Section 2.6: Limits at Infinity; Horizontal Asymptotes

2.6.4.

(a) 2
(b) -2
(c) ∞
(d) $-\infty$
(e) $-\infty$
(f) Horizontal asymptotes: $y = -2$, $y = 2$; Vertical asymptotes: $x = -2$, $x = 0$, $x = 3$

2.6.22. $\frac{1}{3}$ (factor out x from the numerator and pull out the x^2 from inside the square root)

2.6.26.

$$\begin{aligned}
\lim_{x\to-\infty} x + \sqrt{x^2+2x} &= \lim_{x\to-\infty} x + \sqrt{x^2}\sqrt{1+\frac{2}{x}} \\
&= \lim_{x\to-\infty} x + |x|\sqrt{1+\frac{2}{x}} \\
&= \lim_{x\to-\infty} x - x\sqrt{1+\frac{2}{x}} \\
&= \lim_{x\to-\infty} x\left(1-\sqrt{1+\frac{2}{x}}\right) \\
&= \lim_{x\to-\infty} x\left(1-\sqrt{1+\frac{2}{x}}\right)\frac{\left(1+\sqrt{1+\frac{2}{x}}\right)}{\left(1+\sqrt{1+\frac{2}{x}}\right)} \\
&= \lim_{x\to-\infty} x\left(\frac{1^2-\left(\sqrt{1+\frac{2}{x}}\right)^2}{1+\sqrt{1+\frac{2}{x}}}\right) \\
&= \lim_{x\to-\infty} x\left(\frac{1-\left(1+\frac{2}{x}\right)}{1+\sqrt{1+\frac{2}{x}}}\right) \\
&= \lim_{x\to-\infty} x\left(\frac{1-1-\frac{2}{x}}{1+\sqrt{1+\frac{2}{x}}}\right) \\
&= \lim_{x\to-\infty} x\left(\frac{-\frac{2}{x}}{1+\sqrt{1+\frac{2}{x}}}\right) \\
&= \lim_{x\to-\infty} \frac{-2}{1+\sqrt{1+\frac{2}{x}}} \\
&= \frac{-2}{1+1} \\
&= -1
\end{aligned}$$

2.6.32. $-\infty$ (factor out x^3 from the numerator and x^2 from the denominator)

2.6.34. $\tan^{-1}(-\infty) = -\frac{\pi}{2}$ (by continuity of $\tan^{-1}$)

2.6.57. 5 (by the squeeze theorem)

Section 2.7: Derivatives and Rates of Change

2.7.6. $y = x + 4$ ($(y - 3) = (x + 1)$ is also acceptable)

2.7.12.

(a) A runs with constant speed, while B is slow at first and then speeds up
(b) ≈ 8.5 seconds
(c) 9 seconds

2.7.17. $g'(0) < 0 < g'(4) < g'(2) < g'(-2)$

2.7.18.

(a) $y = 4x - 23$ ($y + 3 = 4(x - 5)$ is also acceptable)
(b) $f(4) = 3$, $f'(4) = \frac{1}{4}$

2.7.32. $f(x) = \sqrt[4]{x}$, $a = 16$

2.7.34. $f(x) = \tan(x)$, $a = \frac{\pi}{4}$

2.7.40. $\approx -\frac{5}{6}$ F/min (slope of the red line)

2.7.46.

(a) Rate of bacterias/hour after 5 houts
(b) $f'(10) > f'(5)$ (basically, the more bacteria there are, the more can be produced). But if there's a limited supply of food, we get that $f'(10) < f'(5)$, i.e. bacterias are dying out because of the limited supply

Section 2.8: The derivative as a function

2.8.3.

(a) II
(b) IV
(c) I
(d) III

2.8.21. $f'(t) = 5 - 18t$

2.8.38. -1 (not continuous there); 2 (graph has a kink)

2.8.43.

(a) Acceleration
(b) Velocity
(c) Position

2.8.52. Not differentiable at the integers, because not continuous there; $f'(x) = 0$ for x not an integer, undefined otherwise. Graph looks like the 0-function, except it has holes at the integers.

Section 3.1: Derivatives of polynomials and exponential functions

3.1.11. $y' = -\frac{2}{5}x^{-\frac{7}{5}}$

3.1.13. $V'(r) = 4\pi r^2$, which is the surface area of sphere! What a coincidence - or is it? ;)

3.1.20. $f'(t) = \frac{1}{2\sqrt{t}} + \frac{1}{2t\sqrt{t}} = t^{-\frac{1}{2}} + \frac{1}{2}t^{-\frac{3}{2}}$

3.1.32. $y' = e^{x+1}$

3.1.35. $y' = 4x^3 + 2e^x$, so $y'(0) = 2$, and so the equation of tangent line is $y - 2 = 2(x - 0)$, i.e. $\boxed{y = 2x + 2}$ and equation of normal line is $y - 2 = -\frac{1}{2}(x - 0)$, i.e. $\boxed{y = -\frac{1}{2}x + 2}$ (remember that the normal line still goes through $(0, 2)$, but has slope = the negative reciprocal of the slope of the tangent line)

3.1.49.

(a) $v(t) = s'(t) = 3t^2 - 3$; $a(t) = v'(t) = 6t$
(b) $a(2) = 12$
(c) $v(t) = 0$ if $t = 1$ or $t = -1$, but $t > 0$ (negative time doesn't make sense), so $t = 1$, and $a(1) = 2$

3.1.54. First of all $y' = \frac{3}{2}\sqrt{x}$ and first find a point x where $y'(x) = 3$ (remember that two lines are parallel when their slopes are equal, and the slope of $y = 1+3x$ is 3). So you want $\frac{3}{2}\sqrt{x} = 3$, so $\sqrt{x} = 2$, so $x = 4$. Now all that you need to find out is the slope of the tangent line to the curve at 4. The equation is: $\boxed{y - 8 = 3(x - 4)}$ (because from the above calculation the slope is 3, and the tangent line goes through $(4, f(4)) = (4, 8)$)

Section 3.2: The product and quotient rules

3.2.15. $y' = \frac{2t\left(t^4-3t^2+1\right)-t^2\left(4t^3-6t\right)}{\left(t^4-3t^2+1\right)^2}$

3.2.17. $y' = (2r - 2)e^r + \left(r^2 - 2r\right)e^r = \left(r^2 - 2\right)e^r$

3.2.33. $y'(x) = 2e^x + 2xe^x$, so $y'(0) = 2$, and so the tangent line has equation: $y - 0 = 2(x - 0)$,i.e $\boxed{y = 2x}$, and the normal line has equation: $y - 0 = -\frac{1}{2}(x - 0)$, i.e. $\boxed{y = -\frac{1}{2}x}$

3.2.41. $f'(x) = \frac{2x(1+x)-x^2}{(1+x)^2} = \frac{x^2+2x}{x^2+2x+1}$, so $f''(x) = \frac{(2x+2)(x^2+2x+1)-(x^2+2x)(2x+2)}{(x^2+2x+1)^2}$, and so $f''(1) = \frac{(2+2)(1+2+1)-(1+2)(2+2)}{(1+2+1)^2} = \frac{(4)(4)-(3)(4)}{(4)(4)} = \frac{16-12}{16} = \frac{4}{16} = \boxed{= \frac{1}{4}}$

3.2.47.

(a) $u'(1) = f'(1)g(1) + f(1)g'(1) = (2)(1) + (2)(-1) = \boxed{0}$
(b) $v'(5) = \frac{f'(5)g(5)-f(5)g'(5)}{(g(5))^2} = \frac{\left(-\frac{1}{3}\right)(2)-(3)\left(\frac{2}{3}\right)}{4} = \frac{-\frac{2}{3}-2}{4} = -\frac{8}{12} = \boxed{-\frac{2}{3}}$

3.2.53. (9200)(30593) + (961400)(1400) = 1,345,960,000

Section 3.3: Derivatives of trigonometric functions

3.3.5. $g'(t) = 3t^2\cos(t) - t^3\sin(t)$

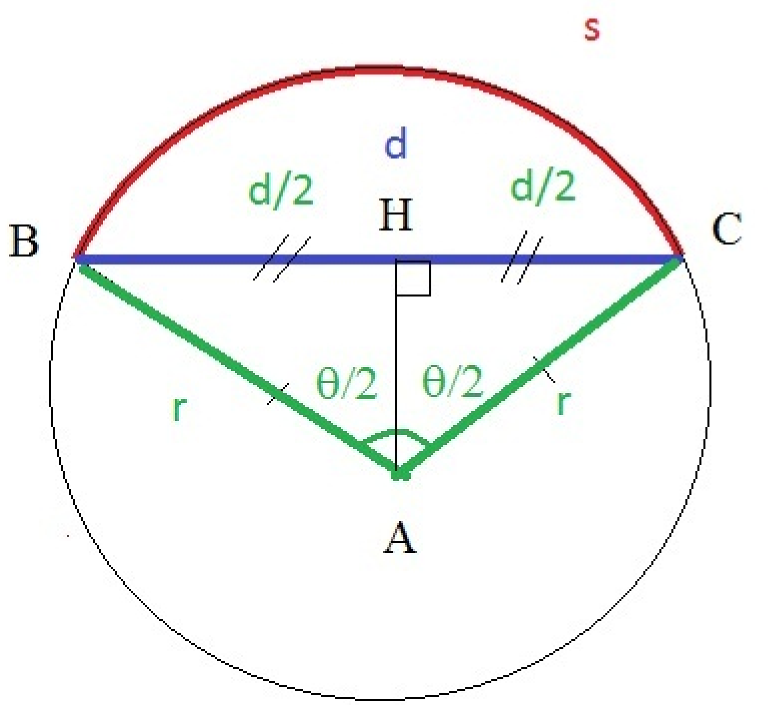
3.3.13. $y' = \frac{\cos(x)x^2 - \sin(x)(2x)}{x^4}$

3.3.37. We have $\sin(\theta) = \frac{x}{10}$, so $x = 10\sin(\theta)$, so $x'(\theta) = 10\cos(\theta)$, and $x'\left(\frac{\pi}{3}\right) = 10\cos\left(\frac{\pi}{3}\right) = \frac{10}{2} = \boxed{5}$

3.3.39. 3 (multiply the fraction by $\frac{3}{3}$ and use the fact that $\lim_{x\to 0}\frac{\sin(3x)}{3x} = 1$)

3.3.40. $\frac{4}{6} = \frac{2}{3}$ (multiply the numerator by $\frac{4}{4}$ and the denominator by $\frac{6}{6}$ and use the facts that $\lim_{x\to 0}\frac{\sin(4x)}{4x} = 1$ and $\lim_{x\to 0}\frac{\sin(6x)}{6x} = 1$)

1A/Math 1A Summer/Solution Bank/Arclength.png



3.3.51. The problem is to calculate:

$$\lim_{\theta\to 0^+} = \frac{s}{d}$$

First of all, let's call the radius of the circle r. Now, let's divide this into three simple sub-problems:

(1) **Calculate s**

But this is not very hard! Either you know about the formula for the length of an arc, or you can easily derive it! Namely, if an angle of 2π radians corresponds to a length of $2\pi r$ (i.e. the circumference of a circle), then an angle of θ radians corresponds to a length of s.
Now, because the length of an arc is proportional to the angle, we have the following equality:

$$\frac{s}{\theta} = \frac{2\pi r}{2\pi} = r$$

So $\boxed{s = \theta r}$

Notice that the use of radians makes this calculation particularly simple!

(2) **Calculate d**

This is a bit harder than the above step, but actually not that bad! Label the triangle in the figure ABC, and let H be the midpoint of BC. Then, the triangle AHB is right in A, and we can use our regular definition of sin to find a relationship between r and d, namely:

$$sin(\angle BAH) = \frac{BH}{AB}$$

But you can easily check that $\angle BAH = \frac{\theta}{2}$, that $BH = \frac{d}{2}$ (because H is the midpoint of BC), and that $AB = r$! Hence, we get:

$$\sin(\frac{\theta}{2}) = \frac{\frac{d}{2}}{r}$$

That is, $\boxed{d = 2rsin(\frac{\theta}{2})}$.

(3) Compute the limit

The last thing we need to do is to calculate the required limit! But this is easy, since we know the values of s and d in terms of r:

$$\lim_{\theta\to 0^+} \frac{s}{d} = \lim_{\theta\to 0^+} \frac{r\theta}{2r\sin(\frac{\theta}{2})} = \lim_{\theta\to 0^+} \frac{\theta}{2\sin(\frac{\theta}{2})} = \lim_{\theta\to 0^+} \frac{\frac{\theta}{2}}{\sin(\frac{\theta}{2})}$$

Finally, let $t = \frac{\theta}{2}$ and notice that, as $\theta \to 0^+$, $t \to 0^+$, then, our limit becomes:

$$\lim_{\theta\to 0^+} \frac{s}{d} = \lim_{t\to 0^+} \frac{t}{\sin(t)} = \lim_{t\to 0^+} \frac{1}{\frac{\sin(t)}{t}} = \frac{\lim_{t\to 0^+} 1}{\lim_{t\to 0^+} \frac{\sin(t)}{t}} = \frac{1}{1} = 1$$

And again, the next-to-last step is justified because both limits (numerator and denominator) exist and the limit of the denominator is nonzero!

Hence, we get:

$\boxed{\lim_{\theta\to 0^+} \frac{s}{d} = 1}$

Section 3.4: The Chain Rule

3.4.9. $F'(x) = \frac{1}{4}(1+2x+x^3)^{-\frac{3}{4}}(2+3x^2)$

3.4.15. $y' = e^{-kx} - kxe^{-kx}$

3.4.39. $f'(t) = \sec^2(e^t) + e^{\tan(t)}\sec^2(t)$

3.4.45. $y' = -\sin(\sqrt{\sin(\tan(\pi x))})\frac{1}{2\sqrt{\sin(\tan(\pi x))}}\cos(\tan(\pi x))\sec^2(\pi x)\pi$

3.4.49. $y' = \alpha e^{\alpha x}\sin(\beta x) + e^{\alpha x}\beta\cos(\beta x)$; $y'' = e^{\alpha x}((\alpha^2-\beta^2)\sin(\beta x) + 2\alpha\beta\cos(\beta x))$

3.4.63.
(a) $h'(1) = f'(g(1))g'(1) = f'(2)g'(1) = 5\times 6 = 30$
(b) $H'(1) = g'(f(1))f'(1) = g'(3)f'(1) = 9\times 4 = 36$

3.4.66.
(a) $h'(2) = f'(f(2))f'(2) = f'(1)f'(2) = -\frac{1}{2}\times 1 = -\frac{1}{2}$
(b) $g'(2) = f'(4)4 = 1\times 4 = 4$

3.4.70.
(a) $f'(x) = g(x^2) + xg'(x^2)2x = g(x^2) + 2x^2g'(x^2)$
(b) $f''(x) = g'(x^2)(2x) + 4xg'(x^2) + 2x^2g''(x^2)2x = 6xg'(x^2) + 4x^3g''(x^2)$

3.4.83. $a(t) = \frac{dv}{dt} = \frac{dv}{ds}\frac{ds}{dt} = \frac{dv}{ds}v(t)$

3.4.84. $V(t) = \frac{4}{3}\pi r^3$, so $\frac{dV}{dt} = \frac{4}{3}\pi 3r^2\frac{dr}{dt} = 4\pi r^2\frac{dr}{dt}$

3.4.90.

$$\begin{aligned}(f(x)[g(x)]^{-1})' &= f'(x)[g(x)]^{-1} + f(x)(-1)[g(x)]^{-2}g'(x)\\ &= f'(x)g(x)[g(x)]^{-2} - f(x)g'(x)[g(x)]^{-2}\\ &= \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}\end{aligned}$$

Section 3.5: Implicit differentiation

3.5.3. $y' = -\frac{y^2}{x^2}$

3.5.19. $y' = \frac{e^y\sin(x)+\cos(xy)y}{e^y\cos(x)-x\cos(xy)}$

3.5.27. $y = x + \frac{1}{2}$

3.5.28. $(y-1)=\sqrt{3}(x+3\sqrt{3})$, or $y=\sqrt{3}x+10$

3.5.30. $y=-2$

3.5.40. First of all, by implicit differentiation:

$$\begin{aligned}\frac{2x}{a^2}+\frac{2yy'}{b^2}=&0\\ y'\left(\frac{2y}{b^2}\right)=&-\frac{2x}{a^2}\\ y'=&-\frac{b^2}{a^2}\frac{2x}{2y}\\ y'=&-\frac{b^2}{a^2}\frac{x}{y}\end{aligned}$$

It follows that the tangent line to the ellipse at (x_0,y_0) has slope $-\frac{b^2}{a^2}\frac{x_0}{y_0}$, and since it goes through (x_0,y_0), its equation is:

$$y-y_0=\left(-\frac{b^2}{a^2}\frac{x_0}{y_0}\right)(x-x_0)$$

And the rest of the problem is just a little algebra!

First of all, by multiplying both sides by a^2y_0, we get:

$$(y-y_0)(a^2y_0)=-b^2x_0(x-x_0)$$

Expanding out, we get:

$$ya^2y_0-a^2(y_0)^2=-b^2x_0x+b^2(x_0)^2$$

Now rearranging, we have:

$$ya^2y_0+b^2x_0x=a^2(y_0)^2+b^2(x_0)^2$$

Now dividing both sides by a^2, we get:

$$yy_0+\frac{b^2}{a^2}x_0x=(y_0)^2+\frac{b^2}{a^2}(x_0)^2$$

And dividing both sides by b^2, we get:

$$\frac{yy_0}{b^2}+\frac{x_0x}{a^2}=\frac{(y_0)^2}{b^2}+\frac{(x_0)^2}{a^2}$$

But now, since (x_0,y_0) is on the ellipse, $\frac{(y_0)^2}{b^2}+\frac{(x_0)^2}{a^2}=1$, we get:

$$\frac{yy_0}{b^2}+\frac{x_0x}{a^2}=1$$

Whence,

$$\frac{x_0 x}{a^2} + \frac{y_0 y}{b^2} = 1$$

Which is what we want!

3.5.46. $y' = \frac{1}{2\arctan(x)} \times \frac{1}{1+x^2}$

3.5.49. $G'(x) = -\frac{x}{\sqrt{1-x^2}} \arccos(x) - 1$

3.5.67.

(a) $f(f^{-1}(x)) = x$, let $y = f^{-1}(x)$, then $f(y) = x$, so $f'(y)y' = 1$, so $y' = \frac{1}{f(y)} = \frac{1}{f(f^{-1}(x))}$

(b) $\frac{3}{2}$

3.5.69. Let's denote the point of intersection between the ellipse and the tangent line by (a, b).

Then, using implicit differentiation, we can show that the slope of the tangent line is

$$-\frac{a}{4b}$$

Now, let K be the altitude of the lamp, our goal is to find K.

Notice that the same tangent line goes through the points $(-5, 0)$ and $(3, K)$, so by the slope formula, we have:

$$\text{Slope} = \frac{K - 0}{3 - (-5)} = \frac{K}{8}$$

In particular, since the slope is also equal to $-\frac{a}{4b}$, we have:

$$\frac{K}{8} = -\frac{a}{4b}$$

So

$$K = -8\frac{a}{4b} = -\frac{2a}{b}$$

So all we really need to do to solve this problem is to find $-\frac{2a}{b}$!

Now we also know that the tangent line goes through the points $(-5, 0)$ and (a, b), so its slope is $\frac{b-0}{a-(-5)} = \frac{b}{a+5}$, but again we know that its slope is also $-\frac{a}{4b}$, and so we get:

$$\frac{b}{a+5} = -\frac{a}{4b}$$

So cross-multiplying, we have $4b^2 = -(a)(a + 5)$, that is $a^2 + 4b^2 = -5a$.

HOWEVER, We also know that (a, b) is on the ellipse, so it satisfies the equation of the ellipse, and so $a^2 + 4b^2 = 5$, whence we get $-5a = 5$, and so $a = -1$.

And plugging $a = -1$ into $a^2 + 4(b)^2 = 5$ and assuming $b > 0$, we get $b = 1$, and so $K = -\frac{2a}{b} = \frac{2}{1} = 2$, and we're done!

Section 3.6: Derivatives of logarithmic functions

3.6.13. $g'(x) = \frac{1}{x\sqrt{x^2-1}}\left(\sqrt{x^2-1} + \frac{x^2}{\sqrt{x^2-1}}\right)$

3.6.21. $y' = 2\log_{10}(\sqrt{x}) + 2x\frac{1}{\ln(10)\sqrt{x}} \times \frac{1}{2\sqrt{x}} = 2\log_{10}(\sqrt{x}) + \frac{1}{\ln(10)}$

3.6.39. $y' = \frac{\sin^2(x)\tan^4(x)}{(x^2+1)^2}\left(2\frac{\cos(x)}{\sin(x)} + 4\frac{\sec^2(x)}{\tan(x)} - \frac{x}{x^2+1}\right)$

Section 3.7: Rates of change in the natural and social sciences

3.7.4.

(a) $f'(t) = e^{-\frac{t}{2}} - \frac{t}{2}e^{-\frac{t}{2}} = e^{-\frac{t}{2}}\left(1 - \frac{t}{2}\right)$

(b) $f'(3) = e^{-\frac{3}{2}}\left(-\frac{1}{2}\right)$

(c) $t = 2$

(d) When $t < 2$

(e) $f(2) - f(0) + f(2) - f(8) = 2e^{-1} - 0 + 2e^{-1} - 8e^{-4} = 4e^{-1} - 8e^{-4}$

(f) The particle is moving to the right between $t = 0$ and $t = 2$, and then to the left from $t = 2$ to $t = 8$.

(g) $f''(t) = -\frac{1}{2}e^{-\frac{t}{2}}\left(1 - \frac{t}{2}\right) + e^{-\frac{t}{2}}\left(-\frac{1}{2}\right) = e^{-\frac{t}{2}}\left(-\frac{1}{2} + \frac{t}{4} - \frac{1}{2}\right) = e^{-\frac{t}{2}}\left(\frac{t}{4} - 1\right)$; $f''(3) = e^{-\frac{3}{2}}\left(-\frac{1}{4}\right)$

(h) Use a calculator

(i) Speeding up when $f''(t) > 0$ and $f'(t) > 0$ or when $f''(t) < 0$ and $f'(t) < 0$. But solving those equations reveals that **none** of the two situations can happen! Hence the particle is constantly slowing down!

3.7.10.

(a) First solve for $v(t) = 0$, where $v(t) = \frac{ds}{dt} = 80 - 32t$, you get $\boxed{t = \frac{80}{32} = \frac{5}{2}}$. So the **maximum height** s^* is $s^* = s(\frac{5}{2}) = 200 - 100 = 100$

(b) To find the time t when the ball is 96ft above the ground, we need to solve the equation $s(t) = 96$, and you get $\boxed{t = 2, 3}$, whence $\boxed{v(2) = 80 - 32 \cdot 2 = 16\frac{ft}{s}}$ and $\boxed{v(3) = 80 - 32 \cdot 3 = -16\frac{ft}{s}}$

3.7.17. $f'(x) = 6x =$ linear density at x. $f'(1) = 6$, $f'(2) = 12$, $f'(3) = 18$. The density is highest at 3 and lowest at 1.

3.7.24. First of all, we know two things, namely $f(0) = 20$ and $f'(0) = 12$.
But by the chain rule:

$$f'(t) = -0.7be^{-0.7t} \frac{-a}{(1+be^{-0.7t})^2} = \frac{0.7abe^{-0.7t}}{(1+be^{-0.7t})^2}$$

So from $f(0) = 20$, we get:

$$\frac{a}{1+b} = 20$$

And from $f'(0) = 12$, we get:

$$\frac{0.7ab}{(1+b)^2} = 12$$

From $\frac{a}{1+b} = 20$, we get $a = 20(1+b)$, and plugging this into the second equation, we get:

$$\begin{aligned} \frac{(0.7)(20)(1+b)b}{(1+b)^2} =& 12 \\ \frac{14b}{1+b} =& 12 \\ 14b =& 12(1+b) \\ 14b =& 12 + 12b \\ 2b =& 12 \\ b = 6 \end{aligned}$$

And so $a = 20(1+6) = 20(7) = 140$.
Therefore, we have $\boxed{a = 140}$ and $\boxed{b = 6}$.

Finally, to find out what happens in the long run, we need to calculate $\lim_{t\to\infty} f(t)$. But notice that $\lim_{t\to\infty} e^{-0.7t} = 0$, and so $\lim_{t\to\infty} f(t) = \frac{a}{1+0} = a = \boxed{140}$.

3.7.29.

(a) $C'(x) = 12 - 0.2x + 0.0015x^2$
(b) $C'(200) = 32$; The cost of producing one more yard of a fabric once 200 yards have been produced
(c) $C(201) - C(200) = 32.2005$, which is pretty close to $C'(200)$

Section 3.8: Exponential growth and decay

3.8.4.

(a) $y(0) = C = 120$
(b) $y(t) = 120e^{\frac{\ln(125)}{6}t} = 120(125)^{\frac{t}{6}} = 120\left(5^{\frac{t}{2}}\right)$
(c) $y(5) = 120 \times 5^{\frac{5}{2}} \approx 6708$
(d) $y'(5) = Ky(5) = \frac{\ln(125)}{6} \times 120 \times 5^{\frac{5}{2}} \approx 5398$
(e) $t = 2\frac{\ln\left(\frac{5000}{3}\right)}{\ln(5)} \approx 9.21$

3.8.9.

(a) $y(t) = 100e^{\ln(\frac{1}{2})\frac{t}{30}} = 100\left(\frac{1}{2}\right)^{\frac{t}{30}}$

(b) $y(100) = 100\left(\frac{1}{2}\right)^{\frac{100}{30}} \approx 9.92$

(c) $t = 30\frac{\ln(\frac{1}{100})}{\ln(\frac{1}{2})} \approx 199.3$

3.8.11. The problem asks about radioactive decay, so as usual, we have $y' = ky$, so $y(t) = Ce^{kt}$. Now we're given two things: First of all, the half-life is $t = 5730$ years, so $y(5730) = \frac{y(0)}{2} = \frac{C}{2}$. Moreover, we know that at a certain time t^* (we **want to find** t^*), $y(t^*) = 0.74y(0) = 0.74C$. Now even though we don't know what C is, we can still solve for t^*.

The following calculation helps us find k:

$$\begin{aligned}
y(5730) =& \frac{C}{2} \\
Ce^{5730k} =& \frac{C}{2} \\
e^{5730k} =& \frac{1}{2} \\
5730k =& \ln(\frac{1}{2}) \\
k =& \frac{\ln(\frac{1}{2})}{5730}
\end{aligned}$$

Whence $y(t) = Ce^{\frac{\ln(\frac{1}{2})}{5730}t} = C(\frac{1}{2})^{\frac{t}{5730}}$

Now we're given that $y(t^*) = 0.74C$, and the following calculation helps us solve for t^*:

$$\begin{aligned}
y(t^*) =& 0.74C \\
C(\frac{1}{2})^{\frac{t^*}{5730}} =& 0.74C \\
(\frac{1}{2})^{\frac{t^*}{5730}} =& 0.74 \\
\frac{t^*}{5730}\ln(\frac{1}{2}) =& \ln(0.74) \\
t^* =& 5730\frac{\ln(0.74)}{\ln(\frac{1}{2})} \\
t^* \approx& 2489
\end{aligned}$$

So $\boxed{t^* \approx 2489 \text{ years}}$ (notice how we didn't even need info about C to figure this out!)

3.8.19.

(a) (i) $3000\left(1 + \frac{0.05}{1}\right)^{(1)(5)} \approx 3828$

(ii) $3000\left(1 + \frac{0.05}{2}\right)^{(2)(5)} \approx 3840$

(iii) $3000\left(1 + \frac{0.05}{12}\right)^{(12)(5)} \approx 3850$

(iv) $3000\left(1+\frac{0.05}{52}\right)^{(52)(5)} \approx 3851.61$
(v) $3000\left(1+\frac{0.05}{365}\right)^{(365)(5)} \approx 3852.01$
(vi) $3000e^{0.05(5)} \approx 3852.08$

(b) $A' = 0.05A,\ A(0) = 3000$

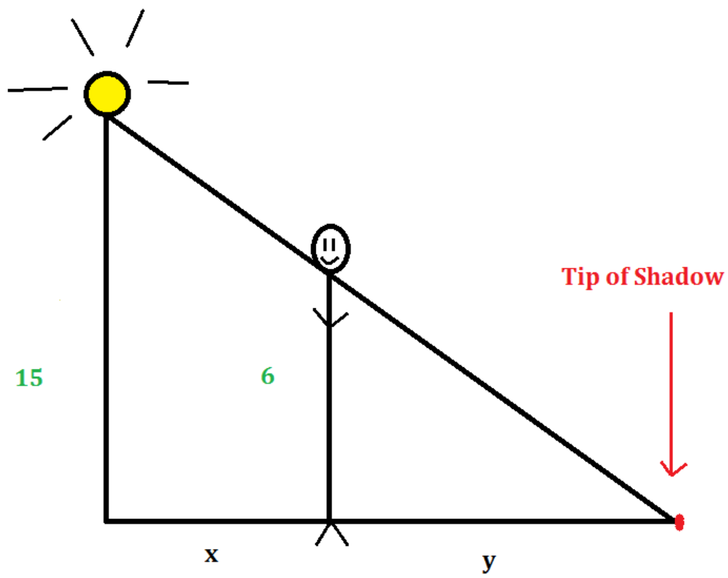
Section 3.9: Related rates

3.9.5. $\boxed{\frac{dh}{dt} = \frac{3}{25\pi}}$ (Use $V = \pi r^2 h$)

3.9.13.

1) First of all, let's draw a picture of the situation, and remember to only label things that are **constant!**

1A/Math 1A Summer/Solution Bank/Street Light.png



Here, x is the distance between the street light and the man, and y is the distance between the man and the shadow. Also, let $z = x + y$, the total length of the shadow.

2) We want to figure out z' when $x = 40$.
3) Looking at the picture, we can use the law of similar triangles to conclude:

$$\frac{y}{x+y} = \frac{6}{15}$$

That is:

$$\begin{aligned} y &= \frac{2}{5}(x+y) \\ \frac{3}{5}y &= \frac{2}{5}x \\ y &= \frac{2}{3}x \end{aligned}$$

It follows that:

$$z = x + y = x + \frac{2}{3}x = \frac{5}{3}x$$

4) Hence $z' = \frac{5}{3}x'$
5) However, we know that $x' = 5$ (because the man is walking with a speed of 5 ft/s).
Hence we get $z' = \frac{5}{3}(5) = \frac{25}{3}$

So $\boxed{z' = \frac{25}{3}}$.

Note: We did not need the fact that $x = 40$!

3.9.15. $\boxed{\frac{dD}{dt} = 65mph}$ (use the pythagorean theorem to conclude $D^2 = x^2 + y^2$)

3.9.27. $\boxed{\frac{dh}{dt} = \frac{6}{5\pi}}$ (use the fact that $V = \frac{\pi}{12}h^3$ because $h = \frac{r}{2}$)

3.9.38.

1) Again, draw a picture of the situation:
Here, x is the distance between P and the beam of light.
2) We want to figure out $\frac{dx}{dt}$ when $x = 1$
3) Looking at the picture, because we have info about the derivative of θ (see below), we use the definition of $\tan(\theta)$:

$$\tan(\theta) = \frac{x}{3}$$

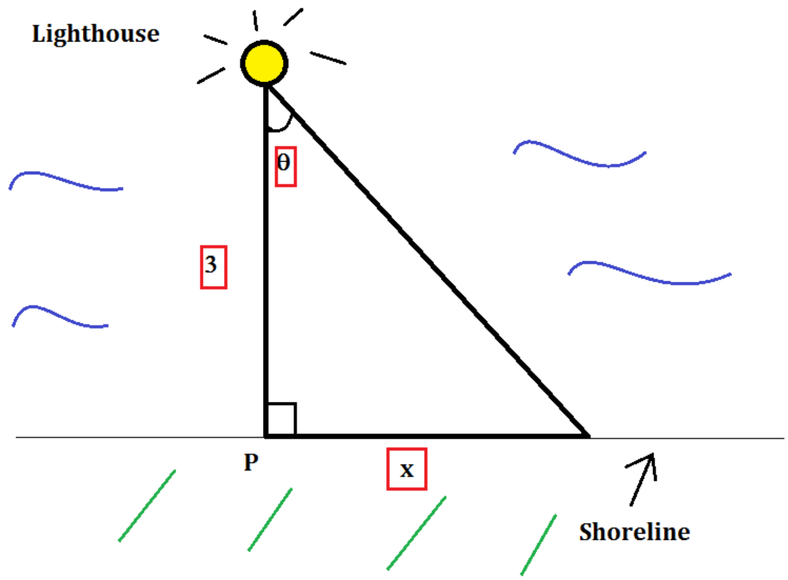
So $x = 3\tan(\theta)$
4) Hence $\frac{dx}{dt} = 3\sec^2(\theta)\frac{d\theta}{dt}$
5) First of all, we actually know what $\frac{d\theta}{dt}$ is! Since the lighthouse makes 4 revolutions per minute and one revolution corresponds to 2π, we know that $\frac{d\theta}{dt} = -8\pi$ (think of speed = distance/time, and here time = 1 minute, and 'distance' = 8π, also you put a minus-sign since x is decreasing!).

Moreover, by drawing the **exact** same picture as above, except with $x = 1$, we can calculate $\sec^2(\theta)$, namely:

$$\sec(\theta) = \frac{hypothenuse}{adjacent} = \frac{\sqrt{10}}{3}$$

And the $\sqrt{10}$ we get from the Pythagorean theorem!

1A/Math 1A Summer/Solution Bank/Lighthouse.png



It follows that $\sec^2(\theta) = \left(\frac{\sqrt{10}}{3}\right)^2 = \frac{10}{9}$.
Now we got all of the info we need to conclude the problem:

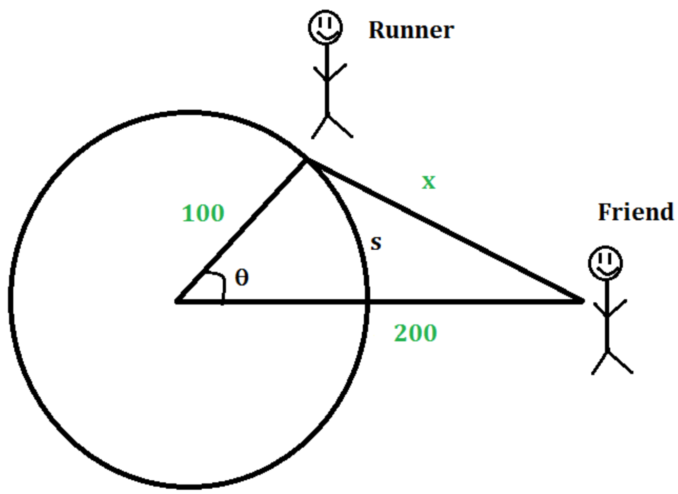
$$\frac{dx}{dt} = 3\sec^2(\theta)\frac{d\theta}{dt}$$
$$\frac{dx}{dt} = 3(\frac{10}{9})(-8\pi)$$
$$\frac{dx}{dt} = -\frac{240\pi}{9}$$
$$\frac{dx}{dt} = -\frac{80\pi}{3}$$

Whence $\boxed{\frac{dx}{dt} = -\frac{80\pi}{3} rad/min}$

3.9.43.

1) As usual, let's draw a picture of the situation:
   Here, x is the distance between the runner and the friend, and s is the length of the arc corresponding to θ
2) We want to figure out $\frac{dx}{dt}$ when $x = 200$

1A/Math 1A Summer/Solution Bank/Runners.png



3) Looking at the picture, it looks like we should use the law of cosines (because we have info about θ and about 2 of the 3 sides of the triangle)

$$x^2 = 100^2 + 200^2 - 2(100)(200)\cos(\theta)$$

In other words:

$$x^2 = 50000 - 40000\cos(\theta)$$

4) Hence $2x\frac{dx}{dt} = 40000\sin(\theta)\frac{d\theta}{dt}$

5) First of all $x = 200$ in this case. Also, by definition of a radian, we know that $s = 100\theta$, whence $\frac{ds}{dt} = 100\frac{d\theta}{dt}$. But we're given that $\frac{ds}{dt} = 7m/s$, so $\frac{d\theta}{dt} = \frac{7}{100} = 0.07$.

So all we got to figure out is $\sin(\theta)$

For this, draw the same picture as above, except you let $x = 200$. And in this case, we use the law of cosines **again**:

$$200^2 = 100^2 + 200^2 - 2(100)(200)\cos(\theta)$$

$$-10000 = -40000\cos(\theta)$$

$$\cos(\theta) = \frac{1}{4}$$

Now you can use either the triangle method to figure out what $\sin(\theta)$ is (all you gotta do is calculate $\sin(\cos^{-1}(\frac{1}{4})$, or, even easier, notice that $\sin(\theta) = \sqrt{1 - \cos^2(\theta)}$ (this works because $\sin(\theta) > 0$ because we assume that θ is between 0 and $\frac{\pi}{2}$. Hence $\sin(\theta) = \sqrt{1 - \frac{1}{16}} = \sqrt{\frac{15}{16}} = \frac{\sqrt{15}}{4}$.

PHEW!!! Now we have all the info we need to solve the problem:

$$\begin{aligned}
2x\frac{dx}{dt} &= 40000\sin(\theta)\frac{d\theta}{dt} \\
2(200)\frac{dx}{dt} &= 40000(\frac{\sqrt{15}}{4})(0.07) \\
400\frac{dx}{dt} &= 700\sqrt{15} \\
\frac{dx}{dt} &= \frac{7}{4}\sqrt{15}
\end{aligned}$$

Hence $\boxed{\frac{dx}{dt} = \frac{7}{4}\sqrt{15}}$

3.9.44. As is usual for related rates problems, let's draw a picture:
Let θ be the angle between the hour hand and the minute hand. Now, what we want to calculate is $D'(\theta)$, where the $'$ indicates differentiation with respect to the time variable.

How can we relate $D(\theta)$ with what we know? This is easy! We know an angle θ and the lengths of AB and AC in the picture, so let's just use the **law of cosines**. We get:

$$BC^2 = AC^2 + AB^2 - 2 \cdot AC \cdot AB \cdot \cos(\theta)$$

That is:

$$D(\theta)^2 = 8^2 + 4^2 - 2 \cdot 8 \cdot 4 \cdot \cos(\theta)$$

Which you can write as:

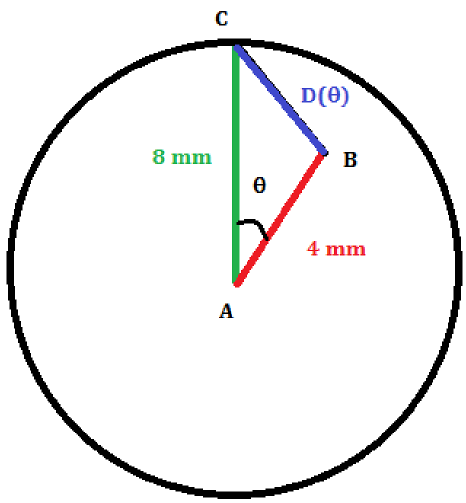
$$D(\theta)^2 = 80 - 64\cos(\theta)$$

Now differentiate with respect to time!
We get:

$$2D(\theta)D'(\theta) = 64\sin(\theta)\frac{d\theta}{dt}$$

And now, all we need to do is to plug in everything we know!

1A/Math 1A Summer/Solution Bank/Clock.png



First of all $\theta = \frac{2\pi}{12} = \frac{\pi}{6}$ (basically, the whole circle corresponds to 2π, and so $\frac{1}{12}$ of the circle corresponds to $\frac{2\pi}{12}$).

In particular, $\boxed{\sin(\theta) = \sin(\frac{\pi}{6}) = \frac{1}{2}}$.

Now, for $d(\theta)$, we use the law of cosines again, this time with the value $\theta = \frac{\pi}{6}$:

$$D(\frac{\pi}{6})^2 = 80 - 64 \cdot \cos(\frac{pi}{6}) = 80 - 64\frac{\sqrt{3}}{2} = 80 - 32\sqrt{3}$$

So, taking square roots, we get $\boxed{d(\frac{\pi}{6}) = \sqrt{80 - 32\sqrt{3}} = 4\sqrt{5 - 2\sqrt{3}} \text{ mm}}$.

Finally, we need to compute $\frac{D\theta}{dt}$. But think about it! In 12 hours, $\theta = 2\pi$, so the speed of θ should be $\frac{2\pi}{12} = \frac{\pi}{6} = -\frac{11\pi}{6}$ rad/h (notice that θ is decreasing, so we wanted a negative answer!)

Finally, we have all our information to get our final answer:

$$\begin{aligned}
2D(\theta)D'(\theta) &= 64\sin(\theta)\frac{D\theta}{dt} \\
2\cdot(4\sqrt{5-2\sqrt{3}})\cdot D'\left(\frac{\pi}{6}\right) &= 64\cdot\frac{1}{2}\cdot\frac{-11\pi}{6} \\
2\cdot(4\sqrt{5-2\sqrt{3}})\cdot D'\left(\frac{\pi}{6}\right) &= 32\cdot\frac{-11\pi}{6} \\
D'\left(\frac{\pi}{6}\right) &= 16\cdot\frac{\frac{-11\pi}{6}}{4\sqrt{5-2\sqrt{3}}} \\
D'\left(\frac{\pi}{6}\right) &= 4\cdot\frac{\frac{-11\pi}{6}}{\sqrt{5-2\sqrt{3}}} \\
D'\left(\frac{\pi}{6}\right) &= -\frac{22\pi}{3\sqrt{5-2\sqrt{3}}} \\
D'\left(\frac{\pi}{6}\right) &\approx -18.55mm/h
\end{aligned}$$

So our final answer is $\boxed{D'\left(\frac{\pi}{6}\right) = -\frac{22\pi}{3\sqrt{5-2\sqrt{3}}}\text{ mm/h} \approx -18.55mm/h}$

Section 3.10: Linear approximations and differentials

3.10.2. $L(x) = x - 1$ (because $\ln(1) = 0$, and $\frac{1}{1} = 1$)

3.10.11.

(a) $dy = (2x\sin(2x) + 2x^2\cos(2x))dx$

(b) $dy = \frac{1}{\sqrt{1+t^2}}\left(\frac{t}{\sqrt{1+t^2}}\right)dt = \frac{t}{1+t^2}dt$

3.10.15.

(a) $dy = \frac{1}{10}e^{\frac{x}{10}}dx$
(b) $dy = \frac{1}{10}(0.1) = 0.01$

3.10.21. $\Delta(y) = y(5) - y(4) = \frac{2}{5} - \frac{2}{4} = -\frac{1}{10} = -0.1$
$dy = -\frac{2}{4^2}(1) = -\frac{1}{8} = -0.125$

3.10.25.

$$(8.06)^{\frac{2}{3}} \approx L(8.06) = 8^{\frac{2}{3}} + \frac{2}{3}(8)^(- \frac{1}{3})(8.06-8) = 4 + \frac{1}{3}(0.06) = 4 + 0.02 = 4.02$$

Here we used the fact that $f(x) = x^{\frac{2}{3}}$ and $a = 8$

3.10.35. $l = 2\pi r = 84$, so $r = \frac{84}{2\pi} = \frac{42}{\pi}$. We know $dl = 0.5$, so $2\pi dr = 0.5$, so $dr = \frac{0.5}{2\pi} = \frac{1}{4\pi}$

(a) $S = 4\pi r^2$, so $dS = 8\pi r dr = 8\pi\frac{42}{\pi}\frac{1}{4\pi} = \frac{84}{\pi}$. Also the relative error is $\frac{dS}{S} = \frac{8\pi r dr}{4\pi r^2} = \frac{2dr}{r} = \frac{1}{2\pi}\times\frac{\pi}{42} = \frac{1}{84} \approx 0.012$
(b) $V = \frac{4}{3}\pi r^3$, so $dV = 4\pi r^2 dr = 4\pi\frac{42^2}{\pi^2}\times\frac{1}{4\pi} = \frac{1764}{\pi^2} \approx 179$. Also the relative error is $\frac{dV}{V} = \frac{4\pi r^2 dr}{\frac{4}{3}\pi r^3} = \frac{3dr}{r} = \frac{\frac{3}{4\pi}}{\frac{42}{\pi}} = \frac{3}{168} = \frac{1}{56} \approx 0.018$

3.10.40. $dF = 4kR^3 dR$, so:

$$\frac{dF}{F} = \frac{4kR^3 dR}{F} = \frac{4kR^3 dR}{F} = \frac{4kR^3 dR}{kR^4} = 4\frac{dR}{R}$$

And when $\frac{dR}{R} = 0.05$, $\frac{dF}{F} = 4(0.05) = 0.2$

Section 3.11: Hyperbolic functions

3.11.9.

$$\cosh(x) + \sinh(x) = \frac{e^x + e^{-x}}{2} + \frac{e^x - e^{-x}}{2} = \frac{e^x + e^{-x} + e^x - e^{-x}}{2} = \frac{2e^x}{2} = e^x$$

3.11.11. Just expand out the right-hand-side and use the fact that $\sinh(x) = \frac{e^x - e^{-x}}{2}$, $\cosh(y) = \frac{e^y - e^{-y}}{2}$, $\cosh(x) = \frac{e^x + e^{-x}}{2}$ and $\sinh(y) = \frac{e^y - e^{-y}}{2}$

3.11.15.

$$2\sinh(x)\cosh(x) = 2\frac{e^x - e^{-x}}{2}\frac{e^x + e^{-x}}{2} = \frac{2}{4}(e^x - e^{-x})(e^x + e^{-x}) = \frac{1}{2}(e^{2x} - e^{-2x}) = \frac{e^{2x} - e^{-2x}}{2} = \sinh(2x)$$

3.11.21. $\sinh(x) = \frac{4}{3}$ (use the fact that $\cosh^2(x) - \sinh^2(x) = 1$ and the fact that $\sinh(x) > 0$ when $x > 0$).

Then you get $\tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{\frac{4}{3}}{\frac{5}{3}} = \frac{4}{5}$, $sech(x) = \frac{1}{\cosh(x)} = \frac{3}{5}$, etc.

3.11.23(a). 1 (use $\tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$ and factor out e^x from the numerator and the denominator)

3.11.26. This is very similar to example 3 on page 257. However, there is a subtle point involved, check out the document 'Subtle point in 3.11.26' for more info!

3.11.29(a)(b). This is also very similar to example 4 on page 257. For (a), use the fact that $\cosh(\cosh^{-1}(x))^2 - \sinh(\cosh^{-1}(x))^2 = 1$ and $\cosh(\cosh^{-1}(x)) = x$. Also, you'll need to fact that $\sinh(\cosh^{-1}(x)) \geq 0$ (and this is because $\cosh^{-1}(x) \geq 0$ by definition, and $\sinh(x) \geq 0$ if $x \geq 0$). (b) is even easier, use the fact that: $1 - \tanh(\tanh^{-1}(x))^2 = sech(\tanh^{-1}(x))^2$ and $\tanh(\tanh^{-1}(x)) = x$.

3.11.31. $f'(x) = \sinh(x) + x\cosh(x) - \sinh(x) = x\cosh(x)$

3.11.39. $y' = \frac{1}{1+\tanh^2(x)}(sech^2(x)) = \frac{sech^2(x)}{1+\tanh^2(x)}$

Section 4.1: Maximum and Minimum Values

4.1.6.

- Absolute maximum: Does not exist (**NOT** 5)
- Absolute minimum: $f(4) = 1$
- Local minimum: $f(2) = 2$, $f(4) = 1$
- Local maximum: $f(3) = 4$, $f(6) = 3$

4.1.38. $-1, 1$ (g' does not exist), 0 (makes $g'(c) = 0$)

4.1.39. 0 (F' does not exist), 4, $\frac{8}{7}$ (makes $F'(c) = 0$)

4.1.51. Candidates: $f(-2) = 11$, $f(3) = 66$ (endpoints), $f(0) = 3$, $f(-1) = 2$, $f(1) = 2$. Absolute maximum: $\boxed{f(3) = 66}$, Absolute minimum: $\boxed{f(-1) = f(1) = 2}$

4.1.57.

1) Evaluate f at the endpoints 0 and $\frac{\pi}{2}$

$$f(0) = 2 + 0 = 2,\ f(\tfrac{\pi}{2}) = 0 + 0 = 0$$

2) Find the critical numbers of f

$$f'(t) = -2\sin(t) + 2\cos(2t)$$

$$\begin{aligned} f'(t) &= 0 \\ -2\sin(t) + 2\cos(2t) &= 0 \\ \sin(t) &= \cos(2t) \end{aligned}$$

Here comes the tricky part! This seems impossible to solve, but ideally we'd like to write the right-hand-side just in terms of $\sin(x)$ in order to have a shot at solving this!

Start with $\cos(2t) = \cos^2(t) - \sin^2(t)$ (the double-angle formula for cos).
Moreover $\cos^2(t) = 1 - \sin^2(t)$ (because $\cos^2(t) + \sin^2(t) = 1$)
So we get $\cos(2t) = 1 - \sin^2(t) - \sin^2(t) = 1 - 2\sin^2(t)$. So our original equation becomes:

$$\sin(t) = 1 - 2\sin^2(t)$$

which you can rewrite as $2\sin^2(t) + \sin(t) - 1 = 0$.

This again looks impossible to solve, but notice that this is just a quadratic equation in $\sin(t)$! So let $X = \sin(t)$, then we get:

$$2X^2 + X - 1 = 0$$

And using the quadratic formula (or your factoring skills), we get:

$$(2X - 1)(X + 1) = 0$$

So $X = \frac{1}{2}$ or $X = -1$. That is, $\sin(t) = \frac{1}{2}$ or $\sin(t) = -1$.
HOWEVER, remember that we're only focusing on $[0, \frac{\pi}{2}]$, so in particular $\sin(t) = \frac{1}{2}$ has only one solution in $[0, \frac{\pi}{2}]$, namely $t = \frac{\pi}{6}$, and $\sin(t) = -1$ has **NO** solution in $[0, \frac{\pi}{2}]$.

It follows that the only critical number of f in $[0, \frac{\pi}{2}]$ is $\boxed{t = \frac{\pi}{6}}$ (there are no numbers where f is not differentiable, so in fact those are all the critical numbers).

And we get $f(\frac{\pi}{6}) = \sqrt{3} + \frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{2}$

3) Compare all the candidates you have:

Our candidates are $f(0) = 2$, $f(\frac{\pi}{2}) = 0$ and $f(\frac{\pi}{6}) = \frac{3\sqrt{3}}{2} \approx 2.59$.

Hence, the absolute minimum of f on $[0, \frac{\pi}{2}]$ is $\boxed{f(\frac{\pi}{2}) = 0}$ (the smallest candidate), and the absolute maximum of f on $[0, \frac{\pi}{2}]$ is $\boxed{f(\frac{\pi}{6}) = \frac{3\sqrt{3}}{2}}$ (the largest candidate)

Section 4.2: The Mean Value Theorem

4.2.6. $f(0) = f(\pi) = 0$, but $f'(x) = \sec^2(x) > 0$. This does not contradict Rolle's Theorem because f is not continuous on $[0, \pi]$ (it is discontinuous at $\frac{\pi}{2}$.

4.2.13. f is differentiable on $[0, 3]$; $c = -\frac{1}{2} \ln\left(\frac{1-e^{-6}}{6}\right)$ (you get this by solving $-2e^{-2c} = \frac{e^{-6}-1}{3}$.

4.2.18. Let $f(x) = 2x - 1 - \sin(x)$

At least one root: $f(0) = -1 < 0$ and $f(\pi) = 2\pi - 1 > 0$ and f is continuous, so by the **Intermediate Value Theorem (IVT)** the equation has at least one root.

At most one root: Suppose there are two roots a and b. Then $f(a) = f(b) = 0$, so by **Rolle's Theorem** there is at least one $c \in (a, b)$ such that $f'(c) = 0$. But $f'(c) = 2 - \cos(c) \neq 0$, which is a contradiction, and hence the equation has at most one root.

4.2.23. By the MVT, $\frac{f(4)-f(1)}{4-1} = f'(c)$ for some c in $(1, 4)$. Solving for $f(4)$ and using $f(1) = 10$, we get $f(4) = 3f'(c) + 10 \geq 6 + 10 = 16$.

4.2.29. This is equivalent to showing:

$$\left| \frac{\sin(a) - \sin(b)}{a - b} \right| \leq 1$$

Which is the same as:

$$\left| \frac{\sin(b) - \sin(a)}{b - a} \right| \leq 1$$

Which is the same as:

$$-1 \leq \frac{\sin(b) - \sin(a)}{b - a} \leq 1$$

But by the MVT applied to $f(x) = \sin(x)$, we get:

$$\frac{\sin(b) - \sin(a)}{b - a} = \cos(c)$$

for some c in (a, b). However, $-1 \leq \cos(c) \leq 1$, and so we're done!

4.2.32. Let $f(x) = 2\sin^{-1}(x)$, $g(x) = \cos^{-1}(1-2x^2)$.

Then $f'(x) = \frac{2}{\sqrt{1-x^2}}$ and:

$$g'(x) = -\frac{-4x}{\sqrt{1-(1-2x^2)^2}} = \frac{4x}{\sqrt{1-1+4x^2-4x^4}} = \frac{4x}{\sqrt{4x^2-4x^4}} = \frac{4x}{2x\sqrt{1-x^2}} = \frac{2}{\sqrt{1-x^2}} = f'(x)$$

(you get this by factoring out $4x^2$ out of the square root. This works because $x \geq 0$)

Hence $f'(x) = g'(x)$, so $f(x) = g(x) + C$

To get C, plug in $x = 0$, so $f(0) = g(0) + C$. But $f(0) = g(0) = 0$, so $C = 0$, whence $\boxed{f(x) = g(x)}$

4.2.34. Let $f(t)$ be the speed at time t. By the MVT with $a = 2:00$ and $b = 2:10$, we get:

$$\frac{f(2:10) - f(2:00)}{2:10 - 2:00} = f'(c)$$

But $2:10 - 2:00 = 10$ minutes $= \frac{1}{6}$ h, so:

$$\frac{50-30}{\frac{1}{6}} = f'(c)$$

Whence: $\boxed{f'(c) = 120}$ for some c between $2:00$ pm and $2:10$ pm. But $f'(c)$ is the acceleration at time c, and so we're done!

4.2.36. This is again a proof by contradiction!

Suppose f has (at least) two fixed points a and b.

Then, by definition of a fixed point, $f(a) = a$, and $f(b) = b$.

However, by the **Mean Value Theorem**, there is a c in (a,b) such that:

$$\frac{f(b) - f(a)}{b-a} = f'(c)$$

Now using the fact that $f(b) = b$ and $f(a) = a$, we get:

$$\frac{b-a}{b-a} = f'(c)$$

So

$$1 = f'(c)$$

That is, $f'(c) = 1$. However, by assumption, $f'(x) \neq 1$ for all x, so in particular setting $x = c$ gives $f'(c) \neq 1$.

but since $f'(c) = 1$, we get $1 \neq 1$, which is a contradiction!

Hence f has at most one fixed point!

Section 4.3: How derivatives affect the shape of a graph

4.3.9.

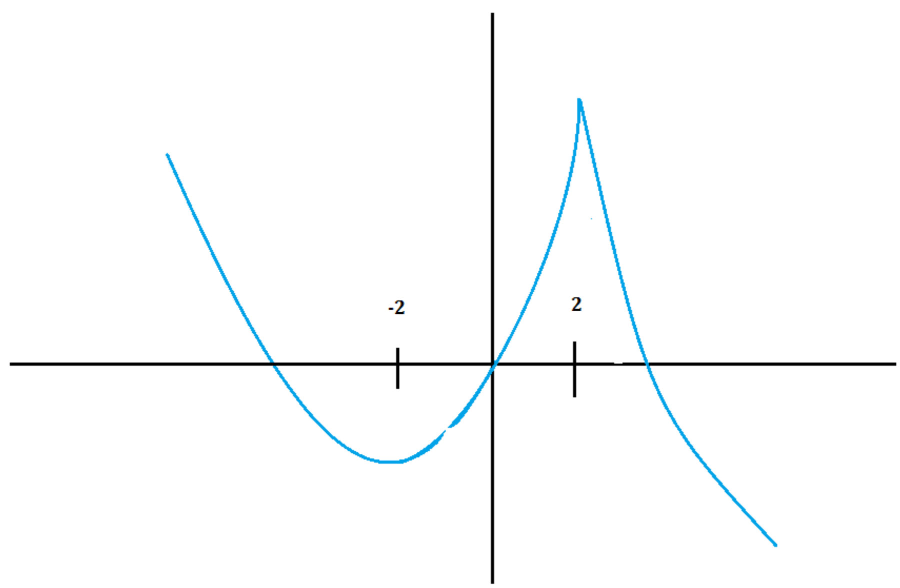
(a) $f'(x) = 6x^2 + 6x - 36 = 6(x-2)(x+3)$; $\nearrow$ on $(-\infty, -3) \cup (2, \infty)$, $\searrow$ on $(-3, 2)$
(b) Local max: $f(-3) = 81$; Local min: $f(2) = -44$
(c) $f''(x) = 12x + 6$; CU on $(-\frac{1}{2}, \infty)$, CD on $(-\infty, \frac{-1}{2})$, IP $(-\frac{1}{2}, f(-0.5) = \frac{37}{2})$

4.3.13.

(a) $f'(x) = \cos(x) - \sin(x)$; $\nearrow$ on $(0, \frac{\pi}{4}) \cup (\frac{5\pi}{4}, \infty)$, $\searrow$ on $(\frac{\pi}{4}, \frac{5\pi}{4})$
(b) Local max: $f(\frac{\pi}{4}) = \sqrt{2}$; Local min: $f(\frac{5\pi}{4}) = -\sqrt{2}$
(c) $f''(x) = -\sin(x) + \cos(x)$; CU on $(\frac{3\pi}{4}, \frac{7\pi}{4})$, CD on $(0, \frac{3\pi}{4}) \cup (\frac{7\pi}{4}, 2\pi)$, IP $(\frac{3\pi}{4}, 0), (\frac{7\pi}{4}, 0)$

4.3.27. A possible graph looks like this:

1A/Math 1A Summer/Solution Bank/Concave-Kink.png



4.3.43.

(a) $f'(\theta) = -2\sin(\theta) - 2\cos(\theta)\sin(\theta) = -2\sin(\theta)(1 + \cos(\theta))$; $\nearrow$ on $(\pi, 2\pi)$, $\searrow$ on $(0, \pi)$
(b) Local min: $f(\pi) = -1$, no local max.

(c) $f''(x) = -2\cos(\theta) + 2\sin^2(\theta) - 2\cos^2(\theta) = -2\cos(\theta) + 2 - 4\cos^2(\theta) = -4(\cos^2(\theta) - \frac{\cos(\theta)}{2} - \frac{1}{2}) = -4(\cos(\theta)+1)(\cos(\theta) - \frac{1}{2})$; CU on $(\frac{\pi}{3}, \frac{5\pi}{3})$, CD on $(0, \frac{\pi}{3}) \cup (\frac{5\pi}{3}, 2\pi)$, IP $(\frac{\pi}{3}, \frac{5}{4}), (\frac{5\pi}{3}, \frac{5}{4})$

4.3.45.

(a) Vertical Asymptotes: $x = -1$, $x = 1$; Horizontal Asymptote: $y = 1$ (at $\pm\infty$)
(b) $f'(x) = \frac{-2x}{(x^2-1)^2}$; $\nearrow$ on $(-\infty, -1) \cup (-1, 0)$, $\searrow$ on $(0, 1) \cup (1, \infty)$ (remember that f is not defined at ± 1)
(c) Local max: $f(0) = 0$; No local min
(d) $f''(x) = \frac{-2(x^2-1)^2+(2x)(2)(x^2-1)(2x)}{(x^2-1)^4} = \frac{6(x^2-1)(x^2+\frac{1}{3})}{(x^2-1)^4}$; CU on $(-\infty, -1) \cup (1, \infty)$, CD on $(-1, 1)$, No inflection points.

4.3.51.

(a) Vertical Asymptote: $x = -1$; Horizontal Asymptote: $y = 1$ (at $\pm\infty$)
(b) $f'(x) = \frac{1}{(x+1)^2} e^{-\frac{1}{x+1}}$; $\nearrow$ on $(-\infty, -1) \cup (-1, \infty)$ (remember f is not defined at -1)
(c) No local max/min
(d) $f''(x) = \frac{-2}{(x+1)^3} e^{-\frac{1}{x+1}} + \frac{1}{(x+1)^4} e^{-\frac{1}{x+1}} = \frac{-2x-1}{(x+1)^4} e^{-\frac{1}{x+1}}$; CU on $(-\infty, -1) \cup (-1, -\frac{1}{2})$, CD on $(-\frac{1}{2}, \infty)$, IP $= (-\frac{1}{2}, \frac{1}{e^2}$.

Section 4.4: L'Hopital's Rule

4.4.3.

(a) No, $-\infty$
(b) Yes, $\infty - \infty$
(c) No, ∞

4.4.4.

(a) Yes, 0^0
(b) No, 0
(c) Yes, 1^∞
(d) Yes, ∞^0
(e) No, ∞
(f) Yes, ∞^0

4.4.11.

$$\lim_{t\to 0} \frac{e^t - 1}{t^3} = \lim_{t\to 0} \frac{e^t}{3t^2} = \lim_{t\to 0} \frac{e^t}{6t} = \lim_{t\to 0} \frac{e^t}{6} = \frac{1}{6}$$

4.4.14.

$$\begin{aligned}\lim_{\theta\to\frac{\pi}{2}} \frac{1-\sin(\theta)}{\csc(\theta)} &= \lim_{\theta\to\frac{\pi}{2}} \frac{-\cos(\theta)}{\frac{-\cos(\theta)}{\sin^2(\theta)}} \text{(l'Hopital's rule)} \\ &= \lim_{\theta\to\frac{\pi}{2}} \sin^2(\theta) \text{(cancelling out)} \\ &= 1\end{aligned}$$

4.4.15.

$$\lim_{x\to\infty}\frac{\ln(x)}{\sqrt{x}}=\lim_{x\to\infty}\frac{\frac{1}{x}}{\frac{1}{2\sqrt{x}}}=\lim_{x\to\infty}\frac{2}{\sqrt{x}}=0$$

4.4.23.

$$\lim_{x\to 0}\frac{\tanh(x)}{\tan(x)}=\lim_{x\to 0}\frac{sech^2(x)}{\sec^2(x)}=1$$

4.4.27.

$$\lim_{x\to 0}\frac{\sin^{-1}(x)}{x}=\lim_{x\to 0}\frac{\frac{1}{\sqrt{1-x^2}}}{1}=1$$

4.4.29.

$$\lim_{x\to 0}\frac{1-\cos(x)}{x^2}=\lim_{x\to 0}\frac{\sin(x)}{2x}=\lim_{x\to 0}\frac{\cos(x)}{2}=\frac{1}{2}$$

4.4.40.

$$\lim_{x\to-\infty}x^2e^x=\lim_{x\to-\infty}\frac{x^2}{e^{-x}}=\lim_{x\to-\infty}\frac{2x}{-e^{-x}}=\lim_{x\to-\infty}\frac{2}{e^{-x}}=0$$

4.4.47.

$$\lim_{x\to 1}\frac{x}{x-1}-\frac{1}{\ln(x)}=\lim_{x\to 1}\frac{x\ln(x)-(x-1)}{(x-1)\ln(x)}=\lim_{x\to 1}\frac{\ln(x)+1-1}{\ln(x)+1-\frac{1}{x}}=\lim_{x\to 1}\frac{\frac{1}{x}}{\frac{1}{x}+\frac{1}{x^2}}=\frac{1}{2}$$

4.4.49.

$$\begin{aligned}\lim_{x\to\infty}\sqrt{x^2+x}-x &=\lim_{x\to\infty}\left(\sqrt{x^2+x}-x\right)\cdot\frac{\sqrt{x^2+x}+x}{\sqrt{x^2+x}+x}\text{(use conjugate form)}\\ &=\lim_{x\to\infty}\frac{x^2+x-x^2}{\sqrt{x^2+x}+x}\\ &=\lim_{x\to\infty}\frac{x}{\sqrt{x^2+x}+x}\\ &=\lim_{x\to\infty}\frac{x}{\sqrt{x^2}\cdot\sqrt{1+\frac{1}{x}}+x}\text{(factor out } x^2 \text{ out of the square root)}\\ &=\lim_{x\to\infty}\frac{x}{x\sqrt{1+\frac{1}{x}}+x}(\sqrt{x^2}=|x|=x,\text{ since } x>0)\\ &=\lim_{x\to\infty}\frac{x}{x\left(\sqrt{1+\frac{1}{x}}+1\right)}\\ &=\lim_{x\to\infty}\frac{1}{\sqrt{1+\frac{1}{x}}+1}\\ &=\frac{1}{\sqrt{1}+1}\\ &=\frac{1}{2}\end{aligned}$$

4.4.51.

$$\lim_{x\to\infty}x-\ln(x)=\lim_{x\to\infty}x(1-\frac{\ln(x)}{x})=\infty\times(1-0)=\infty$$

4.4.56.

1) Let $y = \left(1+\frac{a}{x}\right)^{bx}$
2) $\ln(y) = bx\ln(1+\frac{a}{x})$
3)

$$\lim_{x\to\infty}\ln(y) = \lim_{x\to\infty} bx\ln(1+\frac{a}{x}) = \lim_{x\to\infty}\frac{\ln(1+\frac{a}{x})}{\frac{1}{bx}} = \lim_{x\to\infty}\frac{(\frac{1}{1+\frac{a}{x}})(-\frac{a}{x^2})}{(-\frac{1}{x^2})(\frac{1}{b})} = \lim_{x\to\infty}\frac{ab}{1+\frac{a}{x}} = ab$$

4) So $\lim_{x\to\infty}\left(1+\frac{a}{x}\right)^{bx} = e^{ab}$

4.4.59.

1) Let $y = x^{\frac{1}{x}}$
2) Then $\ln(y) = \frac{\ln(x)}{x}$
3) So $\lim_{x\to\infty}\ln(y) = \lim_{x\to\infty}\frac{\ln(x)}{x} = 0$
4) Hence $\lim_{x\to\infty} x^{\frac{1}{x}} = \lim_{x\to\infty} y = e^0 = 1$

4.4.71.

(a) Here, you want to calculate $\lim_{t\to\infty}$, so **treat t as our x, and leave everything else as a constant!!!!**. In particular, we get:

$$\lim_{t\to\infty} e^{-\frac{ct}{m}} = 0 \qquad \text{because } c > 0 \text{ and } m > 0$$

So, we get:

$$\lim_{t\to\infty} v = \frac{mg}{c}(1-0) = \frac{mg}{c}$$

(b) Here, we let $c\to 0^+$, so **we treat c as our x, and leave everything else as a constant!**.

Then, we have:

$$\begin{aligned}\lim_{c\to 0^+} v = \lim_{c\to 0^+} mg\left(\frac{1-e^{-\frac{ct}{m}}}{c}\right) & \\ = \lim_{c\to 0^+} mg\left(\frac{-\left(-\frac{t}{m}\right)e^{-\frac{ct}{m}}}{1}\right) & \qquad \text{(by l'Hopital's rule)} \\ = mg\left(\frac{\left(\frac{t}{m}\right)e^0}{1}\right) & \\ = mg\left(\frac{\left(\frac{t}{m}\right)1}{1}\right) & \\ = \frac{mgt}{m} & \\ = gt & \end{aligned}$$

Section 4.5: Summary of curve sketching

4.5.11.

D : $\mathbb{R} - \{\pm 3\}$

I : No $x-$intercepts, $y-$intercept: $y = -\frac{1}{9}$

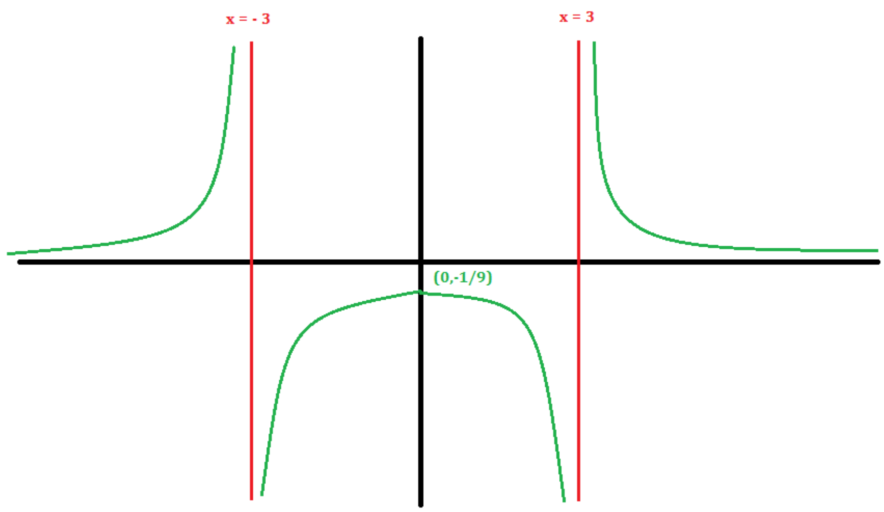
S : f is even

A : Horizontal Asymptote $y = 0$ (at $\pm\infty$), Vertical Asymptotes $x = \pm 3$

I : $f'(x) = -\frac{2x}{(x^2-9)^2}$; f is increasing on $(-\infty, -3) \cup (-3, 0)$ and decreasing on $(0, 3) \cup (3\infty)$. Local maximum of $\frac{-1}{9}$ at 0.

C : $f''(x) = 6\frac{x^2+3}{(x^2-9)^3}$; f is concave up on $(-\infty, -3) \cup (3, \infty)$ and concave down on $(-3, 3)$; No inflection points

1A/Math 1A Summer/Solution Bank/hw10graph1.png



4.5.31.

Note: First of all, f is periodic of period 2π, so we're only focusing on $[0, 2\pi]$.

D : $\mathbb{R}$

I : $x-$intercepts: $x = 0$, $x = 2\pi$ (basically you should get $\sin(x) = 3$, which is impossible), $y-$intercept: $y = 0$

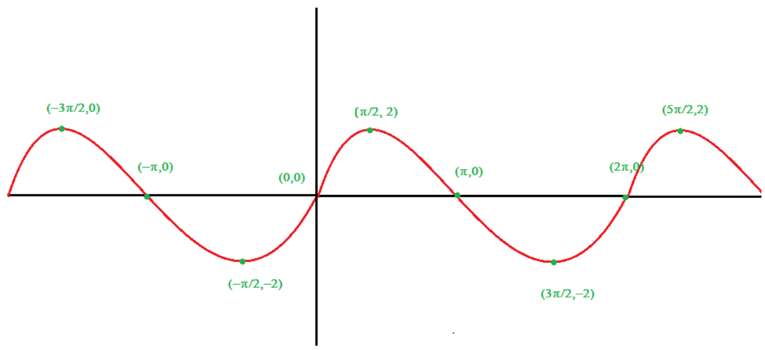
S : Again, f is periodic of period 2π. Also, f is odd.

A : No asymptotes

I : $f'(x) = 3\cos(x) - 3\cos(x)\sin(x) = 3\cos(x)(1 - \sin^2(x)) = 3\cos^3(x)$; Increasing on $(0, \frac{\pi}{2}) \cup (\frac{3\pi}{2}, 2\pi)$; Decreasing on $(\frac{\pi}{2}, \frac{3\pi}{2})$. Local maximum of 2 at $x = \frac{\pi}{2}$. Local minimum of -1 at $x = \frac{3\pi}{2}$.

C : $f''(x) = -9\sin(x)\cos^2(x)$; Concave down on $(0, \pi)$ and Concave up on $(\pi, 2\pi)$. Inflection point $(\pi, 0)$

1A/Math 1A Summer/Solution Bank/hw10graph2.png



4.5.41.

D : $\mathbb{R}$
I : No $x-$intercepts, $y-$intercept: $y = \frac{1}{2}$
S : No symmetries
A : Horizontal Asymptotes: $y = 0$ (at $-\infty$), $y = 1$ (at ∞)
I : $f'(x) = \frac{e^{-x}}{(1+e^{-x})^2} > 0$, so f is decreasing on $\mathbb{R}$
C : $f''(x) = \frac{e^x e^x - 1}{e^x + 1^3}$ (multiply numerator and denominator by $(e^x)^3$ after simplifying), so f is concave down on $(-\infty, 0)$ and concave up on $(0, \infty)$. Inflection point at $(0, \frac{1}{2})$

1A/Math 1A Summer/Solution Bank/hw10graph3.png

4.5.47.

Note: : First of all, $\boxed{f \text{ is periodic of period } 2\pi}$, so from now on we may assume that $x \in [0, 2\pi]$

D : We want $\sin(x) > 0$, so the $\boxed{\text{domain is } (0, \pi)}$

I : $\boxed{\text{No } y\text{−intercepts}}$, x−intercepts: Want $\ln(\sin(x)) = 0$, so $\sin(x) = 1$, so $\boxed{x = \frac{\pi}{2}}$

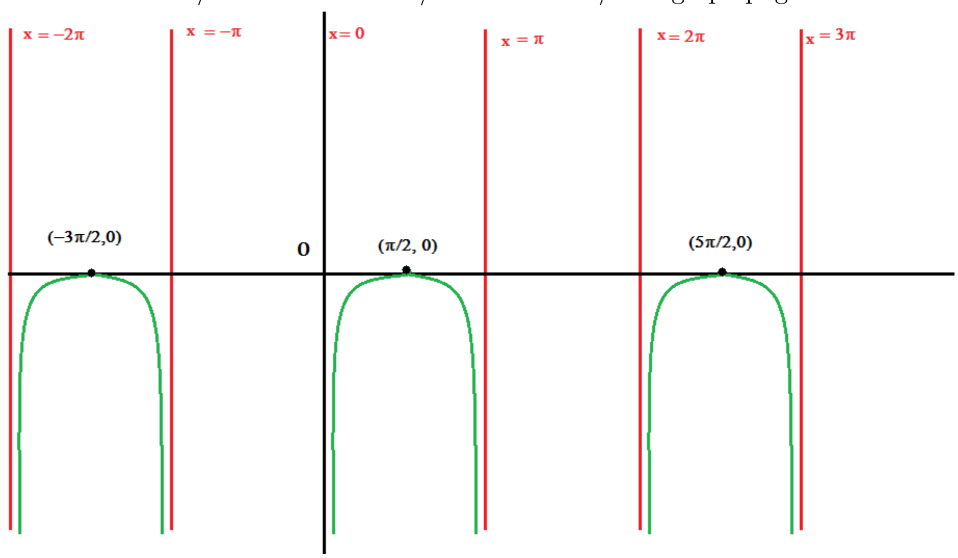
S : Again, f is periodic of period 2π

A : No horizontal/slant asymptotes, but $\lim_{x\to 0^+} \ln(\sin(x)) = \ln(0^+) = -\infty$, so $\boxed{x = 0}$ is a vertical asymptote. Also $\lim_{x\to\pi^-} \ln(\sin(x)) = -\infty$, so $\boxed{x = \pi}$ is also a vertical asymptote.

I : $f'(x) = \frac{\cos(x)}{\sin(x)} = \cot(x)$, then $f'(x) = 0 \Leftrightarrow x = \frac{\pi}{2}$, and using a sign table, we can see that $\boxed{f \text{ is increasing on } (0, \frac{\pi}{2}) \text{ and decreasing on } (\frac{\pi}{2}, \pi)}$. Moreover, $\boxed{f(\frac{\pi}{2}) = \ln(1) = 0}$ is a local maximum of f.

C : $f''(x) = -\csc^2(x) < 0$, so $\boxed{f \text{ is concave down on } (0, \pi)}$.

1A/Math 1A Summer/Solution Bank/hw10graph.png



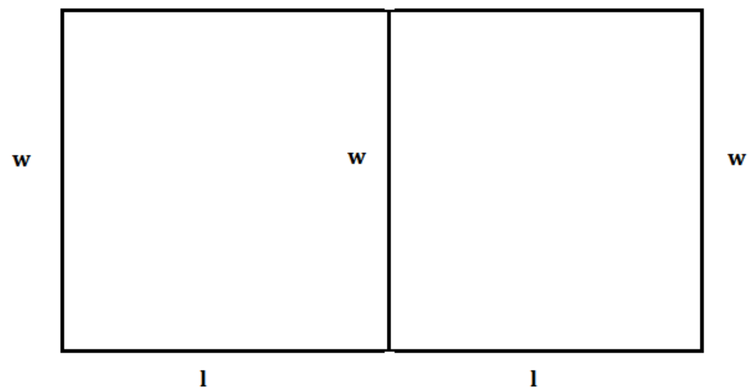
Section 4.7: Optimization Problems

4.7.3.

- Want to minimize $x + y$
- But $xy = 100$, so $y = \frac{100}{x}$, so $x + y = x + \frac{100}{x}$
- Let $f(x) = x + \frac{100}{x}$
- $x > 0$ (x is positive)
- $f'(x) = 0 \Leftrightarrow 1 - \frac{100}{x^2} = 0 \Leftrightarrow x^2 = 100 \Leftrightarrow x = 10$
- By FDTAEV, $x = 100$ is the absolute minimum of f
- Answer: $\boxed{x = 100, y = \frac{100}{100} = 1}$

4.7.11. The picture is as follows:

1A/Math 1A Summer/Solution Bank/Fence.png



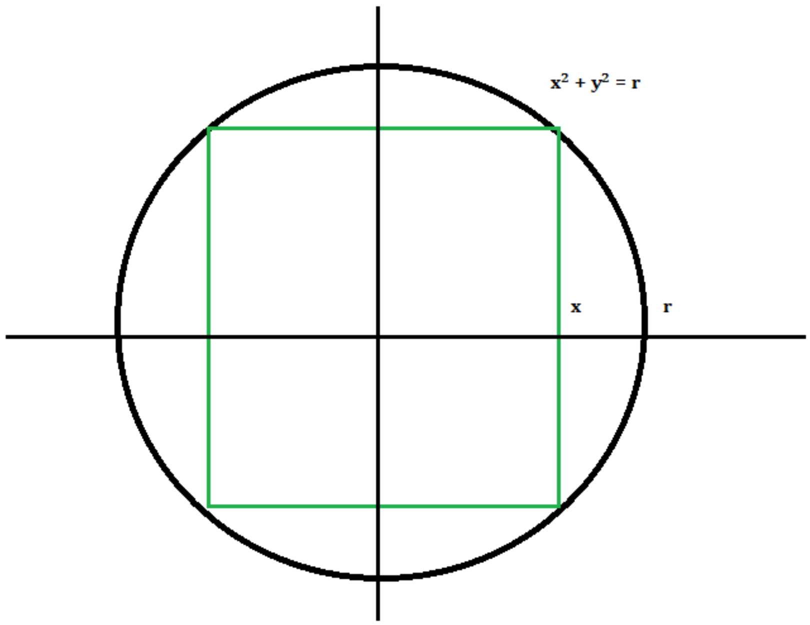
- Want to minimize $3w + 4l$
- But $2lw = 1.5$, so $l = \frac{0.75}{w}$, so $3w + 4l = 3w + \frac{3}{w}$
- Let $f(w) = 3w + \frac{3}{w}$
- $w > 0$
- $f'(x) = 0 \Leftrightarrow 3 - \frac{3}{w^2} = 0 \Leftrightarrow w^2 = 1 \Leftrightarrow w = 1$
- By FDTAEV, $w = 1$ is the absolute minimum of f
- Answer: $\boxed{w = 1,\ 2l = 1.5}$

4.7.19.

- We have $D = \sqrt{(x-1)^2 + y^2}$, so $D^2 = (x-1)^2 + y^2$
- But $y^2 = 4 - 4x^2$, so $D^2 = (x-1)^2 + 4 - 4x^2$
- Let $f(x) = (x-1)^2 + 4 - 4x^2$
- No constraints
- $f'(x) = 2(x-1) - 8x = -6x - 2 = 0 \Leftrightarrow x = -\frac{1}{3}$
- By the FDTAEV, $x = -\frac{1}{3}$ is the maximizer of f.
- Since $y^2 = 4 - 4x^2$, we get $y^2 = 4 - \frac{4}{9} = \frac{32}{9}$, so $y = \pm\sqrt{\frac{32}{9}} = \pm\frac{4\sqrt{2}}{3}$
- Answer: $\boxed{\left(-\frac{1}{3}, -\frac{4\sqrt{2}}{3}\right) \text{ and } \left(-\frac{1}{3}, \frac{4\sqrt{2}}{3}\right)}$

4.7.21. Picture:

1A/Math 1A Summer/Solution Bank/hw10opt1.png



- We have $A = xy$, but the trick here again is to maximize $A^2 = x^2y^2$ (thanks for Huiling Pan for this suggestion!)
- But $x^2 + y^2 = r^2$, so $y^2 = r^2 - x^2$, so $A^2 = x^2(r^2 - x^2) = x^2r^2 - x^4$
- Let $f(x) = x^2r^2 - x^4$
- Constraint $0 \leq x \leq r$ (look at the picture)
- $f'(x) = 2xr^2 - 4x^3 = 0 \Leftrightarrow x = 0$ or $x = \frac{r}{\sqrt{2}}$
- By the closed interval method, $x = \frac{r}{\sqrt{2}}$ is a maximizer of f (basically $f(0) = f(r) = 0$
- Answer: $\boxed{x = \frac{r}{\sqrt{2}},\ y = \sqrt{r^2 - \frac{r^2}{2}} = \frac{r}{\sqrt{2}}}$

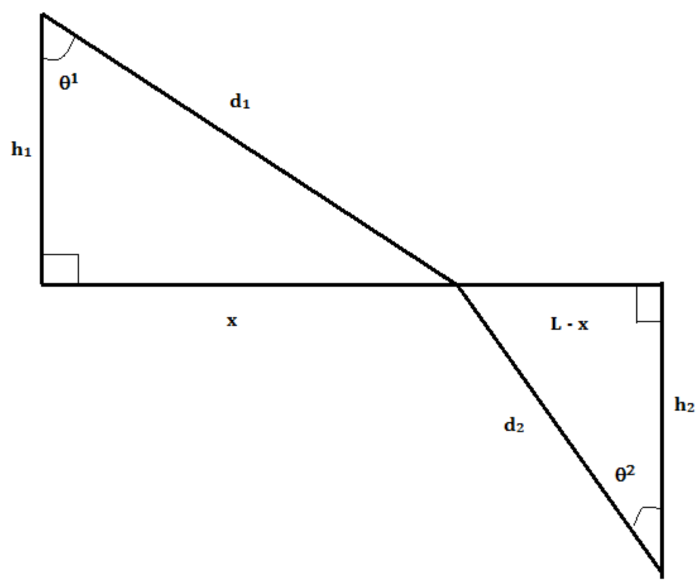
4.7.30.

- Let w be the width of the rectangle, and h the height of the rectangle.
- We have $A = wh + \pi(\frac{w}{2})^2 = wh + \frac{\pi}{4}w^2$, but $w + 2h + 2\pi\frac{w}{2} = 30$, so $2h + \pi w + w = 30$, so $h = \frac{30-(\pi+1)w}{2}$. Hence $A = w(\frac{30-(\pi+1)w}{2}) + \frac{\pi}{4}w^2$
- Let $f(w) = w(\frac{30-(\pi+1)w}{2}) + \frac{\pi}{4}w^2$
- Constraint: $w > 0$
- $f'(w) = 15 - \frac{(\pi+2)}{2}w = 0 \Leftrightarrow w = \frac{30}{\pi+2}$ (there's a big cancellation going on!)
- By FDTAEV, $w = \frac{30}{\pi+2}$ is the maximizer of f
- Answer: $\boxed{w = \frac{30}{\pi+2}, h = \frac{15}{\pi+2}}$

4.7.53. (*a*) $c'(x) = \frac{C'(x)x - C(x)}{x^2}$. When c is at its minimum, $c'(x) = 0$, so $C'(x)x - C(x) = 0$, so $C'(x) = \frac{C(x)}{x} = c(x)$, so $\boxed{C'(x) = c(x)}$, i.e. marginal cost equals the average cost!

4.7.63. (thank you Brianna Grado-White for the solution to this problem!)
The picture is as follows:

1A/Math 1A Summer/Solution Bank/hw10opt2.png



Here, h_1 and h_2 and L are fixed, but x varies.
Now the total time taken is $t = t_1 + t_2 = \frac{d_1}{v_1} + \frac{d_2}{v_2}$.

Now, by the Pythagorean theorem: $d_1 = \sqrt{x^2 + h_1^2}$ and $d_2 = \sqrt{(L-x)^2 + h_2^2}$, so we get:

$$t(x) = \frac{\sqrt{x^2 + h_1^2}}{v_1} + \frac{\sqrt{(L-x)^2 + h_2^2}}{v_2}$$

And

$$t'(x) = \frac{x}{v_1\sqrt{x^2 + h_1^2}} + \frac{x-L}{v_2\sqrt{(L-x)^2 + h_2^2}} = \frac{x}{v_1 d_1} + \frac{x-L}{v_2 d_2}$$

Setting $t'(x) = 0$ and cross-multiplying, we get:

$$v_1 d_1 (L-x) = v_2 d_2 x$$

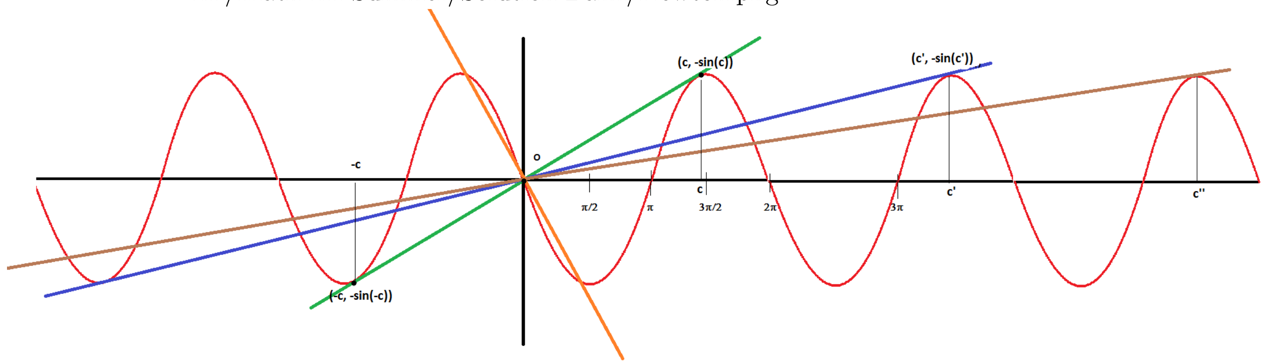
So, by definition of $\sin(\theta_1)$ and $\sin(\theta_2)$), we get:

$$\frac{v_1}{v_2} = \frac{d_2 x}{(L-x)d_1} = \frac{\frac{x}{d_1}}{\frac{L-x}{d_2}} = \frac{\sin(\theta_1)}{\sin(\theta_2)}$$

Section 4.8: Newton's method

4.8.38. As usual, a good picture is the key to solving the problem:

1A/Math 1A Summer/Solution Bank/Newton.png



Note: This picture is very complete. It is meant to illustrate all the points I am making below.

Here, we are given a lot of information, so let's try to tackle this problem one step at a time!

First, let's calculate the equation of *any* tangent line to the graph of $f(x) = -sin(x)$ that goes through $(0, 0)$. Later, we will be worrying about finding the one that has the largest slope.

By definition of the derivative, the tangent line to the graph of f at c has slope $f'(c) = -\cos(c)$, so any such tangent line that **also** goes through $(0, 0)$ has equation: $y - 0 = -\cos(c)(x - 0)$, i.e. $y = -\cos(c)x$. Finally, we know that the tangent line goes through $(c, -\sin(c))$ (i.e. goes through the graph of f at c), so we get: $-\sin(c) = -\cos(c) \cdot c$, i.e. $\tan(c) = c$.

So any tangent line at c with the above properties must solve $\tan(c) = c$, i.e. $\tan(c) - c = 0$.

So what we really need to do is to approximate the zero of the function $g(x) = \tan(x) - x$. Now this looks like a Newton's method problem! But remember, that for Newton's method, we need to find a good initial guess, and **here** is where we use the information that the tangent line must have largest slope!

First of all, notice that $f(x) = -\sin(x)$ is odd, so the graph is symmetric about the origin! If you look at the picture above, you'll see that the tangent line to the graph at c is the same as the tangent line at $-c$. This means we can restrict ourselves to the right-hand-side of the picture, i.e. $c \geq 0$! (if you don't understand this argument, don't worry, it's just a simplification)

And if you look at the picture again, you'll notice that if your initial guess is between 0 and $\frac{\pi}{2}$, your successive approximations will got to 0. And you don't want that because the slope of the tangent line at 0 is -1 (which is not the greatest slope). The same problem arises with the initial guess between $\frac{3\pi}{2}$ and 2π (the approximations go to 2π)

Finally, notice that when c gets larger and larger, the tangent line at c has smaller and smaller slope (see picture: The brown line has a smaller slope than the blue line, which has a smaller slope than the green line), so you'd like your initial guess not to be too large. In particular, we don't want the initial guess to be larger than $\frac{3\pi}{2}$!

From this analysis, we conclude that any initial guess between $\frac{\pi}{2}$ and $\frac{3\pi}{2}$ is good enough! (also look at the picture above: the green tangent line seems to be the winner here)

For example, with $x_0 = 4.5$ (or you could try $x_0 = \pi$), we get the successive approximations (remember that you are applying Newton's method to $g(x) = \tan(x) - x$, **NOT** $f(x)$!):

$$\begin{aligned} x_0 &= 4.5 \\ x_1 &= 4.49361390 \\ x_2 &= 4.49340966 \\ x_3 &= 4.49340946 \\ x_4 &= 4.49340946 \end{aligned}$$

And so, our approximation is: $\boxed{c \approx 4.49340946}$. And hence the largest **slope** is approximately equal to $\boxed{-\cos(4.49340946) \approx 0.2172336}$ (because $f'(c) = -\cos(c)$).

To summarize:

- Draw a picture
- Derive the function that you want to apply Newton's method to (i.e. $g(x) = \tan(x) - x$)
- Argue that your intial approximation must be between $\frac{\pi}{2}$ and $\frac{3\pi}{2}$ (**YOU NEED TO JUSTIFY THIS PART!**, maybe not as precise as I did, but there needs to be some justification
- Apply Newton's method to g with initial approximation between $\frac{\pi}{2}$ and $\frac{3\pi}{2}$ (4 would work)

Section 4.9: Antiderivatives

4.9.7. $F(x) = 5\frac{x^{\frac{5}{4}}}{\frac{5}{4}} - 7\frac{x^{\frac{7}{4}}}{\frac{7}{4}} + C = 4x^{\frac{5}{4}} - 4x^{\frac{7}{4}} + C$

4.9.24. $f'(x) = 2x + \frac{1}{4}x^4 + \frac{1}{7}x^7 + A$, so $f(x) = x^2 + \frac{1}{20}x^5 + \frac{1}{56}x^8 + Ax + B$

4.9.33. $f(x) = -2\sin(t) + \tan(t) + C$, but $4 = f(\frac{\pi}{3}) = -\sqrt{3} + \sqrt{3} + C = C$, so $f(x) = -2\sin(t) + \tan(t) + 4$

4.9.39. If $f''(\theta) = \sin(\theta) + \cos(\theta)$, then $f'(\theta) = -\cos(\theta) + \sin(\theta) + C$.
$f'(0) = 4$, so $-1 + 0 + C = 4$, so $C = 5$.
Hence $f'(\theta) = -\cos(\theta) + \sin(\theta) + 5$.
Hence $f(\theta) = -\sin(\theta) - \cos(\theta) + 5\theta + C'$.
$f(0) = 3$, so $-0 - 1 + 0 + C' = 3$, so $C' = 4$.
Hence $\boxed{f(\theta) = -\sin(\theta) - \cos(\theta) + 5\theta + 4}$

4.9.61. $a(t) = 10\sin(t) + 3\cos(t)$, so $v(t) = -10\cos(t) + 3\sin(t) + A$, so $s(t) = -10\sin(t) - 3\cos(t) + At + B$

Now, $s(0) = 0$, but $s(0) = -10(0) - 3(1) + A(0) + B$, so $-3 + B = 0$, so $B = 3$

So $s(t) = -10\sin(t) - 3\cos(t) + At + 3$

Moreover, $s(2\pi) = 12$, but $s(2\pi) = -10(0) - 3(1) + A(2\pi) + 3 = A(2\pi)$, so $A(2\pi) = 12$, so $A = \frac{12}{2\pi} = \frac{6}{\pi}$

So altogether, you get: $\boxed{s(t) = -10\sin(t) - 3\cos(t) + \frac{6}{\pi}t + 3}$

4.9.74. First of all, the acceleration of the car is $a(t) = -16$, so $v(t) = -16t + C$. We want to find $v(0) = C$, so once we find C, we're done!

Let t^* be the time when the car comes to a stop.
Then $v(t^*) = 0$, so $-16t^* + C = 0$, so $C = 16t^*$. So once we find t^*, we're done!

Now we know that $s(t^*) - s(0) = 200$, but $s(t) = -8t^2 + Ct + C'$, so $200 = -8(t^*)^2 + Ct^* + C' + 0 - C(0) - C' = -8(t^*)^2 + 16t^*t^* = 8(t^*)^2$, so $8(t^*)^2 = 200$, so $(t^*)^2 = 25$ so $t^* = 5$ (assuming time is positive)

Whence $\boxed{v(0) = C = 16t^* = 80}$

Section 5.1: Areas and Distances

5.1.2.
(a) (i) $\Delta x = 2$, so

$$L_6 = f(0)(2)+f(2)(2)+f(4)(2)+f(6)(2)+f(8)(2)+f(10)(2) = 18+\frac{52}{3}+\frac{50}{3}+\frac{44}{3}+12+8 = \frac{260}{3} \approx 86.67$$

(ii)

$$R_6 = f(2)(2)+f(4)(2)+f(6)(2)+f(8)(2)+f(10)(2)+f(12)(2) = \frac{52}{3}+\frac{50}{3}+\frac{44}{3}+12+8+2 = \frac{212}{3} \approx 70.67$$

(iii)

$$M_6 = f(1)(2)+f(3)(2)+f(5)(2)+f(7)(2)+f(9)(2)+f(11)(2) = 18+17+15+13+10+\frac{16}{3} = \frac{235}{3} \approx 78.33$$

(b) Overestimate
(c) Underestimate
(d) M_6 (just right, does not overshoot, like L_6, but not undershoot either, like R_6)

5.1.5.

(a) If $n = 3$, then $\Delta x = 1$, and if $n = 6$, $\Delta x = \frac{1}{2}$, so:

$$R_3 = f(0)(1) + f(1)(1) + f(2)(1) = 1 + 2 + 5 = 8$$

$$\begin{aligned} R_6 =& f(-0.5)(0.5) + f(0)(0.5) + f(0.5)(0.5) + f(1)(0.5) + f(1.5)(0.5) + f(2)(0.5) \\ =& 1.25(0.5) + 1(0.5) + 1.25(0.5) + 2(0.5) + 3.25(0.5) + 5(0.5) \\ =& 6.875 \end{aligned}$$

(b)

$$L_3 = f(-1)(1) + f(0)(1) + f(1)(1) = 2 + 1 + 2 = 5$$

$$\begin{aligned} L_6 =& f(-1)(0.5) + f(-0.5)(0.5) + f(0)(0.5) + f(0.5)(0.5) + f(1)(0.5) + f(1.5)(0.5) \\ =& 2(0.5) + 1.25(0.5) + 1(0.5) + 1.25(0.5) + 2(0.5) + 3.25(0.5) \\ =& 5.375 \end{aligned}$$

(c)

$$M_3 = f(-0.5)(1) + f(0.5)(1) + f(1.5)(1) = 5.75$$

$$M_6 = f(-0.75)(0.5)+f(-0.25)(0.5)+f(0.25)(0.5)+f(0.75)(0.5)+f(1.25)(0.5)+f(1.75)(0.5) = 5.9375$$

(d) M_6

0.1. **5.1.11.** Here $n = 6$ and $\Delta x = 0.5$

$$\begin{aligned} L_6 =& v(0)(0.5) + v(0.5)(0.5) + v(1)(0.5) + v(1.5)(0.5) + v(2)(0.5) + v(2.5)(0.5) \\ =& 0(0.5) + 6.2(0.5) + 10.8(0.5) + 14.9(0.5) + 18.1(0.5) + 19.4(0.5) \\ =& 34.7 \end{aligned}$$

$$\begin{aligned} R_6 =& v(0.5)(0.5) + v(1)(0.5) + v(1.5)(0.5) + v(2)(0.5) + v(2.5)(0.5) + v(3)(0.5) \\ =& 6.2(0.5) + 10.8(0.5) + 14.9(0.5) + 18.1(0.5) + 19.4(0.5) + 20.2(0.5) \\ =& 44.8 \end{aligned}$$

5.1.15. The midpoint sum seems to best approximate the area:

$$M_6 = v(0.5)(1)+v(1.5)(1)+v(2.5)(1)+v(3.5)(1)+v(4.5)(1)+v(5.5)(1) = 50+40+30+18+10+5 = 153ft$$

5.1.19. $\Delta x = \frac{\pi}{2n}$, $x_i = \frac{\pi i}{2n}$, so:

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} \Delta x f(x_i) = \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{\pi}{2n}\right) \frac{\pi i}{2n} \cos(\frac{\pi i}{2n})$$

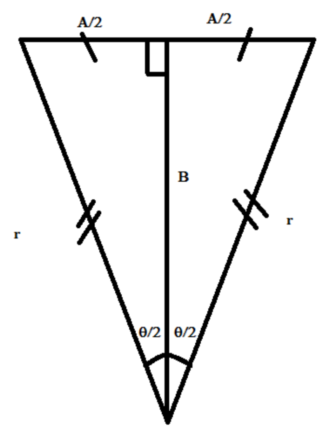
5.1.21. The area under the curve of $f(x) = \tan(x)$ from 0 to $\frac{\pi}{4}$

5.1.26.

(a) We are given that the polygon is made out of n congruent triangles, so $A_n = n \cdot T$, where T is the area of each triangle. So all we need to find is T.

Here again, a picture tells a thousand words, so by drawing the picture of such a triangle, we can figure out its area:

1A/Math 1A Summer/Solution Bank/Polygon.png



Using the picture, you'll notice that:

$$T = \frac{1}{2} \cdot A \cdot B = \frac{A}{2} B$$

And we can divide the triangle into two right triangles, and hence use trigonometry to calculate $\frac{A}{2}$ and B! Here, $\theta = \frac{2\pi}{n}$, the central angle!

We get:

$$\cos\left(\frac{\theta}{2}\right) = \frac{B}{r}$$

$$B = r \cdot \cos\left(\frac{\theta}{2}\right)$$

$$\sin\left(\frac{\theta}{2}\right) = \frac{\frac{A}{2}}{r}$$

$$\frac{A}{2} = r \cdot \sin\left(\frac{\theta}{2}\right)$$

And so, we get:

$$T = \frac{A}{2} \cdot B = r \cdot \sin\left(\frac{\theta}{2}\right) \cdot r \cdot \cos\left(\frac{\theta}{2}\right) = r^2 \cdot \sin\left(\frac{\theta}{2}\right) \cos\left(\frac{\theta}{2}\right) = r^2 \frac{1}{2} \sin\left(2 \cdot \frac{\theta}{2}\right) = \frac{1}{2} r^2 \sin(\theta) = \frac{1}{2} r^2 \sin\left(\frac{2\pi}{n}\right)$$

Here, we used the fact that, in general, $2\sin(x)\cos(x) = \sin(2x)$, so $\sin(x)\cos(x) = \frac{1}{2}\sin(2x)$.

And so, we have:

$$A_n = n \cdot T = n \cdot \frac{1}{2}r^2 \sin\left(\frac{2\pi}{n}\right) = \frac{1}{2}nr^2 \sin\left(\frac{2\pi}{n}\right)$$

(b) Actually, the hint tells us that we don't even have to use l'Hopital's rule, but rather the rule that:

$$\lim_{x\to 0} \frac{\sin(x)}{x} = 1$$

We have that:

$$\begin{aligned}
\lim_{n\to\infty} A_n &= \lim_{n\to\infty} \frac{1}{2}r^2 n \sin\left(\frac{2\pi}{n}\right) \\
&= \frac{1}{2}r^2 \lim_{n\to\infty} n \sin\left(\frac{2\pi}{n}\right) \\
&= \frac{1}{2}r^2 \lim_{n\to\infty} \frac{\sin\left(\frac{2\pi}{n}\right)}{\frac{1}{n}} \\
&= \frac{1}{2}r^2 \lim_{n\to\infty} \frac{2\pi \sin\left(\frac{2\pi}{n}\right)}{\frac{2\pi}{n}} \\
&= 2\pi \cdot \frac{1}{2}r^2 \lim_{n\to\infty} \frac{\sin\left(\frac{2\pi}{n}\right)}{\frac{2\pi}{n}} \\
&= \pi r^2 \lim_{x\to 0} \frac{\sin(x)}{x} \qquad \left(x = \frac{2\pi}{n}\right) \\
&= \pi r^2 (1) \\
&= \pi r^2
\end{aligned}$$

The basic idea is that, if we have an indeterminate form of the form "$0 \cdot \infty$", we rewrite ∞ as $\frac{1}{0}$, or we rewrite 0 as $\frac{1}{\infty}$. Here, for example, we wrote $n = \frac{1}{\frac{1}{n}}$ in order to apply the hint in the problem!

And hooray, you just proved that the formula for the area of a circle of radius r is πr^2. But actually, you didn't, because trigonometry, which you used in (a), relies heavily on this formula!

Section 5.2: The definite integral

5.2.11. $\Delta x = \frac{1}{5} = 0.2$, so:

$$\begin{aligned}\int_0^1 \sin(x^2)dx \approx & f(0.1)(0.2) + f(0.3)(0.2) + f(0.5)(0.2) + f(0.7)(0.2) + f(0.9)(0.2)\\ = & \sin(0.01)(0.2) + \sin(0.09)(0.2) + \sin(0.25)(0.2) + \sin(0.49)(0.2) + \sin(0.81)(0.2)\\ = & \approx 0.3789\end{aligned}$$

5.2.18. $\int_\pi^{2\pi} \frac{\cos(x)}{x} dx$

5.2.21. Here $\Delta x = \frac{6}{n}$ and $x_i = -1 + \frac{6i}{n}$.

$$\begin{aligned}\int_{-1}^5 (1+3x)dx =& \lim_{n\to\infty} \sum_{i=1}^n \Delta x(1+3x_i)\\ =& \lim_{n\to\infty} \sum_{i=1}^n \frac{6}{n}(1+3(-1+\frac{6i}{n}))\\ =& \lim_{n\to\infty} \sum_{i=1}^n \frac{6}{n}(-2+\frac{18i}{n})\\ =& \lim_{n\to\infty} \sum_{i=1}^n \frac{-12}{n} + \frac{108i}{n^2})\\ =& \lim_{n\to\infty} \sum_{i=1}^n \frac{-12}{n} + \sum_{i=1}^n \frac{108i}{n^2})\\ =& \lim_{n\to\infty} \frac{-12}{n}(n) + \frac{108}{n^2}\sum_{i=1}^n i)\\ =& \lim_{n\to\infty} -12 + \frac{108}{n^2}\frac{n(n+1)}{2}\\ =& -12 + \frac{108}{2}\\ =& -12 + 54\\ =& 42\end{aligned}$$

And you thought 42 was **not** the answer to everything =) !

5.2.23. Here $\Delta x = \frac{2}{n}$ and $x_i = \frac{2i}{n}$

$$\begin{aligned}
\int_0^2 (2-x^2)dx &= \lim_{n\to\infty} \sum_{i=1}^{n} \Delta x (2-(x_i)^2) \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{2}{n}(2-\frac{4i^2}{n^2}) \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{4}{n} - \frac{8i^2}{n^3} \\
&= \lim_{n\to\infty} \frac{4}{n}\sum_{i=1}^{n} 1 - \frac{8}{n^3}\sum_{i=1}^{n} i^2 \\
&= \lim_{n\to\infty} \frac{4}{n}(n) - \frac{8}{n^3}\frac{n(n+1)(2n+1)}{6} \\
&= 4 - \frac{8}{6}2 \\
&= \frac{4}{3}
\end{aligned}$$

5.2.30. $\Delta x = \frac{9}{n}$, so $x_i = 1 + \frac{9i}{n}$, and so:

$$\int_1^{10} x - 4\ln(x)dx = \lim_{n\to\infty}\sum_{i=1}^{n} \Delta(x)(x_i - 4\ln(x_i)) = \lim_{n\to\infty}\sum_{i=1}^{n} \frac{9}{n}\left(\left(1+\frac{9i}{n}\right) - 4\ln\left(1+\frac{9i}{n}\right)\right)$$

5.2.34.

(a) 4 (the area of the large triangle)
(b) -2π (minus the area of the semicircle)
(c) $4 - 2\pi + \frac{1}{2} = \frac{9}{2} - 2\pi$ (the area of the large triangle minus the area of the semicircle plus the area of the small triangle)

5.2.36. 2π (it's the area of a semicircle with radius 2)

5.2.43. $\int_0^1 5 - 6x^2 dx = 5\int_0^1 1dx - 6\int_0^1 x^2 dx = 5 - 6\frac{1}{3} = 5 - 2 = 3$

5.2.44. $\int_1^3 2e^x - 1dx = 2\int_1^3 e^x dx - \int_1^3 1dx = 2(e^3 - e) - 2$

5.2.47.

$$\int_{-2}^{2} f(x)dx + \int_{2}^{5} f(x)dx - \int_{-2}^{-1} f(x)dx = \int_{-2}^{5} f(x)dx - \int_{-2}^{-1} f(x)dx = \int_{-2}^{5} f(x)dx + \int_{-1}^{-2} f(x)dx = \int_{-1}^{5} f(x)$$

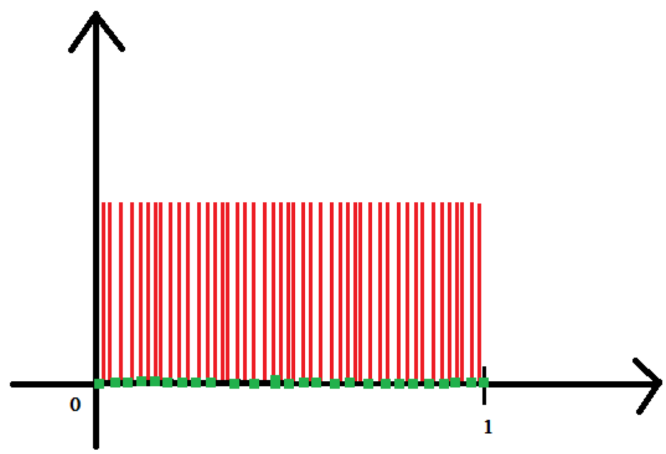
5.2.51. $m \leq f(x) \leq M$, so integrating from 0 to 2, we get $2m \leq \int_0^2 f(x)dx \leq 2M$

5.2.52. On $[0,1]$, $x^2 \leq x$, so $1 + x^2 \leq 1 + x$, so $\sqrt{1+x^2} \leq \sqrt{1+x}$, so integrating from 0 to 1, we get $\int_0^1 \sqrt{1+x^2}dx \leq \int_0^1 \sqrt{1+x}dx$

5.2.67. As usual, let $f(x)$ be as in the problem, $a = 0$, $b = 1$, $x_i = \frac{i}{n}$, and $\Delta(x) = \frac{1}{n}$.

First, let's draw a picture of what's going on:

1A/Math 1A Summer/Solution Bank/Nonintegrable.png



In the picture above, the green dots represent where $f(x) = 0$ and the red lines represent where $f(x) = 1$. This problem is unlike the problem above! In this case, the function does not blow up to infinity, but it can't make up its mind! We need to somehow use this fact in order to show that f is not integrable!

But even though this problem is different, the general strategy is almost the same. f integrable means that no matter how we choose the x_i^*, we get the same answer! So to show that something is **NOT** integrable, we have to pick two different sets of points x_i^* and y_i^* that give us two different answers!

And here is where we use the fact that f looks the way it does. Namely, let x_i^* be your favorite rational number in $[x_{i-1}, x_i]$ and y_i^* your favorite irrational number in $[x_{i-1}, x_i]$! For example (you don't have to write this, but it's better if you do!), you can choose:

$$x_i^* = x_i = \frac{1}{n}$$
$$y_i^* = \frac{i}{\sqrt{2}n}$$

And you can check that $x_i^* \in [x_{i-1}, x_i]$, and $y_i^* \in [x_{i-1}, x_i]$.

But the point is that x_i^* is rational, and so $f(x_i^*) = 0$ by definition of f, and thus the Riemann sum equals to:

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta(x) = \lim_{n\to\infty} \sum_{i=1}^{n} 0 \cdot \frac{1}{n} = 0$$

And y_i^* is rational, and so $f(y_i^*) = 1$ by definition of f, and thus the Riemann sum equals to:

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(y_i^*)\,\Delta(x) = \lim_{n\to\infty} \sum_{i=1}^{n} 1 \cdot \frac{1}{n} = \lim_{n\to\infty} \frac{n}{n} = \lim_{n\to\infty} 1 = 1$$

And so we get two different answers (even though we're supposed to get the same answer if f were integrable)!!! Which shows that f is not integrable on $[0, 1]$!

5.2.68. Let $f(x) = \frac{1}{x}$, $a = 0$, $b = 1$. Then $x_i = \frac{i}{n}$ and $\Delta(x) = \frac{1}{n}$.

How can we show that something is not integrable? The main point is: **Given** n **we need to CHOOSE a set of points** $x_i^* \in [x_{i-1}, x_i]$ **that 'fails'** (whatever that might mean). As discussed in section, the following choice is a good one:

$$\begin{aligned} x_1^* &= \frac{1}{n^2} \\ x_i^* &= x_i \qquad \text{for } i \geq 2 \end{aligned}$$

This works **BECAUSE** $x_1^* \in [x_0, x_1] = [0, \frac{1}{n}]$ and $x_i^* \in [x_{i-1}, x_i]$ (for $i \geq 2$). **Always check this on the exam!**

Because then, we have:

$$\sum_{i=1}^{n} f(x_i^*)\,\Delta(x) \geq f(x_1^*)\,\Delta(x) = \frac{1}{\frac{1}{n^2}} \cdot \frac{1}{n} = \frac{n^2}{n} = n$$

Here, we use the fact that every term in the sum is positive, so the sum is greater than its first term $f(x_1^*)\,\Delta(x)$. Also, $\Delta(x) = \frac{1}{n}$.

And now, if we let $n \to \infty$, the right-hand-side goes to ∞, and so by comparison,

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta(x) = \infty$$

So with this choice of x_i^*, things have gone awry! The Riemann sum 'blows' up to infinity, and so f is not integrable over $[0, 1]$. The point is: **if a function is integrable, then its integral has to be finite.**

Other solution:

Some people wrote up another solution, which is also pretty clever!

Basically, let $x_i^* = x_i = \frac{i}{n}$ $(1 \leq i \leq n)$, which is in $[x_{i-1}, x_i]$.

Then:

$$\sum_{i=1}^{n} f(x_i^*)\Delta(x) = \sum_{i=1}^{n} f\left(\frac{i}{n}\right)\cdot\frac{1}{n} = \sum_{i=1}^{n}\left(\frac{n}{i}\right)\cdot\frac{1}{n} = \sum_{i=1}^{n}\frac{1}{i}$$

However, some of you might know that:

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{i} = \infty$$

It's ok if you don't know this, you're not even supposed to know this because it's covered in Math 1B! (that's why I don't know how many points you would actually get on the exam for this answer...)

And thus:

$$\sum_{i=1}^{n} f(x_i^*)\Delta(x) = \lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{i} = \infty$$

And hence f is not integrable on $[0,1]$ **WARNING:** Note that you **CANNOT**

just say that f is not integrable because it has a vertical asymptote at $x = 0$! For example, the function $g(x) = \frac{1}{\sqrt{x}}$ has a vertical asymptote at $x = 0$, but:

$$\int_0^1 \frac{1}{\sqrt{x}}dx = \left[2\sqrt{x}\right]_0^1 = 2$$

Because $2\sqrt{x}$ is an antiderivative of $\frac{1}{\sqrt{x}}$

Section 5.3: The Fundamental Theorem of Calculus

5.3.4.

(a) $g(-3) = 0$, $g(3) = 0$
(b) $g(-2) \approx 2$, $g(-1) \approx 4$, $g(0) \approx 6$
(c) $(-3, 0)$
(d) 0
(f) $g'(x) = f(x)$

5.3.7. $\frac{1}{x^3+1}$

5.3.15. $\sec^2(x)\sqrt{\tan(x) + \sqrt{\tan(x)}}$

5.3.17. $3\frac{(1-3x)^3}{1+(1-3x)^2}$

5.3.25. $\frac{7}{8}$ (antiderivative is $-\frac{1}{t^3}$)

5.3.31. 1 (antiderivative is $\tan(t)$)

5.3.35. $\frac{\ln(9)}{2} = \ln(3)$ (antiderivative is $\ln(|x|)$)

5.3.41. $1 + (-1) = 0$ (split up the integral into $\int_0^{\frac{\pi}{2}} \sin(x)dx + \int_{\frac{\pi}{2}}^{\pi} \cos(x)dx$)

5.3.43. $\frac{1}{x^4}$ is discontinuous at 0 (the FTC applies only to continuous functions)

5.3.54. $g'(x) = 2x\frac{1}{\sqrt{2+x^8}} - \sec^2(x)\frac{1}{\sqrt{2+\sec^8(x)}}$

Section 5.4: Indefinite Integrals and the Net Change Theorem

5.4.10. $\frac{v^4}{6} + \frac{2}{3}v^2 + \frac{2}{3} + C$

5.4.12. $\frac{x^3}{3} + x + \tan^{-1}(x) + C$

5.4.13. $-\cos(x) + \cosh(x) + C$

5.4.25. 52 (antiderivative is $3x^3 + 3x^2 + x + \frac{1}{9}$)

5.4.37. $1 + \frac{\pi}{4}$ (antiderivative is $x + \tan(x)$)

5.4.47. $\frac{4}{3}$ (antiderivative is $y^2 - \frac{y^3}{3}$)

5.4.52. The bee population after 15 weeks

5.4.58.

(a) $s(3) - s(5) = -\frac{10}{3}$ (antiderivative is $\frac{t^3}{3} - t^2 - 8t$)
(b) $(s(1) - s(4)) + (s(6) - s(4)) = 18 + \frac{44}{3} = \frac{98}{3}$

5.4.59.

(a) $v(t) = t^2 + 4t + 5$
(b) $s(10) - s(0) = \frac{1750}{3}$ (antiderivative is $\frac{t^3}{3} + 2t^2 + 5t$)

5.4.61. $\frac{140}{3}$ (antiderivative is $9x + \frac{4}{3}x^{\frac{3}{2}}$, and $a = 0$, $b = 4$)

5.4.62. 1800 (antiderivative is $200t - 2t^2$, $a = 0$, $b = 10$)

Section 5.5: The substitution rule

5.5.7. $\frac{1}{2}\cos(x^2)$ ($u = x^2$, $du = 2xdx$)

5.5.31. $-\frac{1}{\sin(x)}$ ($u = \sin(x)$, $du = \cos(x)dx$)

5.5.39. $\frac{1}{3}\sec^3(x)$ ($u = \sec(x)$, $du = \sec(x)\tan(x)$)

5.5.46. $\frac{1}{5}(x^2+1)^{\frac{5}{2}} - \frac{1}{3}(x^2+1)^{\frac{3}{2}}$ ($u = x^2 + 1$, $du = 2xdx$, $x^2 = u - 1$)

5.5.59. $e - \sqrt{e}$ ($u = \frac{1}{x}$, $du = -\frac{1}{x^2}dx$, $a = 1$, $b = \frac{1}{2}$)

5.5.73. $0 + 6\pi$ (the first integral is 0 because the function is an odd function, or use $u = 4 - x^2$, $du = -2xdx$, $a = 0$, $b = 0$, and the second integral represents the area of a semicircle with radius 2)

5.5.88.

(a) For the first integral, let $u = \cos(x)$, then $du = -\sin(x)dx = -\sqrt{1-u^2}dx$, so the first integral becomes $\int_1^0 \frac{f(u)}{-\sqrt{1-u^2}}du = \int_0^1 \frac{f(u)}{\sqrt{1-u^2}}du$. For the second integral, let $u = \sin(x)$, then $du = \cos(x)dx = \sqrt{1-u^2}dx$, so the second integral becomes $\int_0^1 \frac{f(u)}{\sqrt{1-u^2}}du$, and it is now clear that both integrals are equal!

(b) By (a) with $f(x) = x^2$ (for the first step), and the fact that $\sin^2(x) = 1 - \cos^2(x)$, we get:

$$\int_0^{\frac{\pi}{2}} \cos^2(x)dx = \int_0^{\frac{\pi}{2}} \sin^2(x)dx = \int_0^{\frac{\pi}{2}} 1dx - \int_0^{\frac{\pi}{2}} \cos^2(x)dx = \frac{\pi}{2} - \int_0^{\frac{\pi}{2}} \cos^2(x)dx$$

Solving for $\int_0^2 \cos^2(x)dx$, we get: $\boxed{\int_0^2 \cos^2(x)dx = \frac{\pi}{4}}$, and hence $\boxed{\int_0^2 \sin^2(x)dx = \frac{\pi}{4}}$ (by (a))

Section 6.1: Areas between curves

6.1.1. $\int_0^4 (5x - x^2) - xdx = \int_0^4 4x - x^2 dx = \boxed{\frac{32}{3}}$

6.1.3. $\int_{-1}^1 e^y - (y^2 - 2)dy = \boxed{e - e^{-1} + \frac{10}{3}}$

6.1.13. $\int_{-3}^3 (12 - x^2) - (x^2 - 6)dx = \int_{-3}^3 18 - 2x^2 dx = = \boxed{72}$
(points of intersection are $x = \pm 3$)

6.1.21. $\int_{-1}^1 (1 - y^2) - (y^2 - 1)dy = \int_{-1}^1 2 - 2y^2 dy = = \boxed{\frac{8}{3}}$
(points of intersection are $y = \pm 1$)

6.1.40. $\int_{-\frac{1}{2}}^{\frac{1}{2}} 1 - |y| - 2y^2 dy = \int_{-\frac{1}{2}}^0 1 + y - 2y^2 dy + \int_0^{\frac{1}{2}} 1 - y - 2y^2 dy = -\frac{7}{24} + \frac{7}{24} = \boxed{\frac{7}{6}}$.
(to find the points of intersection, solve $2y^2 = 1 - |y|$, and split up into the two cases $y \geq 0$ and $y < 0$). Also, it might help to notice that your function is even, so you really only care about the case where $y \geq 0$.

6.1.41. Here $n = 5$, and $D \approx 2(f(1) + f(3) + f(5) + f(7) + f(9)) = 2(2 + 6 + 9 + 11 + 12) = \boxed{80}$, where $f(x) = v_K - v_C$ (notice that $v_K \geq v_C$ throughout the race!)

6.1.49. The first region has area equal to $\int_0^b 2\sqrt{y}dy = \frac{4}{3}b^{\frac{3}{2}}$ (notice that we're integrating with respect to y, and $y = x^2 \Leftrightarrow y = \pm\sqrt{x}$. Also, draw a picture to see why we have an extra factor of 2 in the integral). The second region has area equal to $\int_b^4 2\sqrt{y}dy = -\frac{4}{3}b^{\frac{3}{2}} + \frac{32}{3}$, so to solve for b, we need to set those two areas equal:

$$\frac{4}{3}b^{\frac{3}{2}} = -\frac{4}{3}b^{\frac{3}{2}} + \frac{32}{3} \Leftrightarrow \frac{8}{3}b^{\frac{3}{2}} = \frac{32}{3} \Leftrightarrow b^{\frac{3}{2}} = 4 \Leftrightarrow b = 4^{\frac{2}{3}}$$

Section 6.2: Volumes

6.2.3. Disk method, $K = 0$, $\int_1^2 \pi \left(\frac{1}{x}\right)^2 dx = \boxed{\frac{\pi}{2}}$

6.2.6. Disk method, $K = 0$, $x = e^y$, so $\int_1^2 \pi (e^y)^2 dy = \int_1^2 \pi (e^{2y}) dy = \boxed{\frac{\pi}{2}(e^4 - e^2)}$

6.2.13. Washer method, $K = 1$, Outer $= (3) - 1 = 2$, Inner $= (1 + \sec^2(x)) - 1 = \sec^2(x)$, Points of intersection $\pm\frac{\pi}{3}$, so:

$$\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \pi(2^2 - \sec^2(x))dx = \pi(4\frac{2\pi}{3} - \tan(\frac{\pi}{3}) + \tan(\frac{-\pi}{3})) = \pi(\frac{8\pi}{3} - 2\sqrt{3}) = 2\pi\left(\frac{4}{3}\pi - \sqrt{3}\right)$$

6.2.17. Washer method, $K = -1$, and notice $y = x^2 \Leftrightarrow x = \sqrt{y}$ (in this case $x \geq 0$), Outer $= \sqrt{y} - (-1) = \sqrt{y} + 1$, Inner $= y^2 - (-1) = y^2 + 1$, Point of intersection $y = 0$ and $y = 1$, so:

$$\int_0^1 \pi(\sqrt{y} + 1)^2 - (y^2 + 1)^2 dy = \frac{29\pi}{30}$$

6.2.49. Disk method, $K = 0$, $\int_0^h \pi \left(r - \frac{r}{h}x\right)^2 dx = \boxed{\frac{\pi}{3}r^2 h}$ (the point is to rotate the usual cone by $90°$ so that its height lies on the $x-$axis, and the base disk lies on the $y-$axis., and this it's easy to use the disk method!)

6.2.51. Disk method, $K = 0$, $\int_{r-h}^r \pi(\sqrt{r^2 - y^2})^2 dy = \int_{r-h}^r \pi(r^2 - y^2) dy \boxed{\pi h^2 \left(r - \frac{1}{3}h\right)}$ (use the fact that $x^2 + y^2 = r^2$, and solve for y)

6.2.57. $A(x) = \frac{1}{2}L^2 = \frac{1}{2}(\frac{b}{\sqrt{2}})^2 = \frac{1}{4}b^2 = \frac{1}{4}(2y)^2 = y^2 = \frac{36-9x^2}{4} = 9 - \frac{9}{4}x^2$ (here L is the length of a side of the triangle, and $b = 2y$ is the hypotenuse) so $V = \int_{-2}^2 \left(9 - \frac{9}{4}x^2\right) dx = \boxed{24}$ (you get the endpoints by setting $y = 0$ in $9x^2 + 4y^2 = 36$)

6.2.67. The point is to draw a very good picture! Make one sphere have center $(0, -\frac{r}{2})$ in the $xy-$plane and the other one have center $(0, \frac{r}{2})$. Then the volume is really the volume of two pieces of equal volume, let's focus on $x \geq 0$ only! Then, using the disk method, you get:

$$V = 2\int_0^{\frac{r}{2}} \pi \left(\sqrt{r^2 - \left(x + \frac{r}{2}\right)^2}\right)^2 dx = 2\pi \int_0^{\frac{r}{2}} r^2 - \left(x + \frac{r}{2}\right) dx = \frac{5\pi r^3}{12}$$

(here we used the fact that $(x + \frac{r}{2})^2 + y^2 = r^2$, and solved for y. This looks a bit strange, but remember that your height is really on the left sphere, not on the right one!)

6.2.70. This is **much** easier with the shell method of section 6.3. Here $K = 0$, $f(x) = \sqrt{R^2 - x^2}$ (since $x^2 + y^2 = R^2$), and so $\int_r^R 2\pi x\sqrt{R^2 - x^2}dx = \boxed{\frac{2\pi}{3}\left(R^2 - r^2\right)^{\frac{3}{2}}}$ (use the substitution $u = R^2 - x^2$)

Section 6.3: Volumes by cylindrical shells

6.3.2. $\int_0^{\sqrt{\pi}} 2\pi x \sin(x^2)dx = \boxed{2\pi}$ (use the substitution $u = x^2$)

6.3.13. Shell method: $K = 0$, $|y - 0| = y$, Outer $= 2$, Inner $= 1+(y-2)^2$, Points of intersection $y = 1$, $y = 3$, so $\int_1^3 2\pi y(2-(1+(y-2)^2))dy = \int_1^3 2\pi y(1-(y-2)^2))dy = \boxed{\frac{16\pi}{3}}$.

6.3.15. Shell method: $K = 2$, $|x - 2| = 2 - x$, Outer $= x^4$, Inner $= 0$, $\int_0^1 2\pi(2 - x)(x^4)dx = \boxed{\frac{7\pi}{15}}$

6.3.19. Shell method: $K = 1$, $|y - 1| = 1 - y$, Outer $= 1$, Inner $= \sqrt[3]{y}$, $\int_0^1 2\pi(1 - y)(1 - \sqrt[3]{y})dy = \boxed{\frac{5\pi}{14}}$

6.3.44. Shell method: $K = 0$, $|x| = x$, Outer $= \sqrt{r^2 - (x - R)^2}$ (use the fact that $(x-R)^2 + y^2 = r^2$), Innter $= -\sqrt{r^2 - (x - R)^2}$, so $\int_{R-r}^{R+r} 2\pi x 2\sqrt{r^2 - (x - R)^2}dx = \boxed{\pi^2 R r^2}$ (use the substitution $u = x - R$, and remember what you did in 5.5.73)

6.3.46. Shell method: $K = 0$, $|x| = x$, Outer $= 2\sqrt{R^2 - x^2}$ (use the fact that $x^2 + y^2 = R^2$), Inner $= 0$,

$$\int_r^R 2\pi x(2\sqrt{R^2 - x^2})dx = \frac{4\pi}{3}(R^2 - r^2)^{\frac{3}{2}} = \frac{4\pi}{3}\left(\left(\frac{h}{2}\right)^2\right)^{\frac{3}{2}} = \frac{4\pi}{3}\frac{h^3}{8} = \frac{\pi h^3}{6}$$

(use the substitution $u = R^2 - r^2$, and the fact that $r^2 + (\frac{h}{2})^2 = R^2$ by the Pythagorean theorem)